ON THE ESSENTIAL LOGICAL STRUCTURE OF INTER-UNIVERSAL TEICHMÜLLER THEORY IN TERMS OF LOGICAL AND “$\wedge$”/LOGICAL OR “$\vee$” RELATIONS:

REPORT ON THE OCCASION OF THE PUBLICATION OF THE FOUR MAIN PAPERS ON INTER-UNIVERSAL TEICHMÜLLER THEORY

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Abstract. The main goal of the present paper is to give a detailed exposition of the **essential logical structure** of **inter-universal Teichmüller theory** from the point of view of the *Boolean operators* — such as the *logical AND “$\wedge$” and logical OR “$\vee$” operators* — of propositional calculus. This *essential logical structure* of inter-universal Teichmüller theory may be summarized **symbolically** as follows:

$$\begin{array}{lcl} A \wedge B \;=\; A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ \ldots) & \Longrightarrow & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ \ldots \ \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \ \ldots) \\ & & \qquad\qquad\qquad \vdots \end{array}$$

— where

- the “$\dot{\vee}$” denotes the Boolean operator *exclusive-OR*, i.e., *“XOR”*;
- A, B, B_1, B_2, B'_1, B'_2, denote various propositions;
- the **logical AND “$\wedge$’s”** correspond to the **Θ-link** of inter-universal Teichmüller theory and are closely related to the **multiplicative** structures of the rings that appear in the domain and codomain of the Θ-link;
- the **logical XOR “$\dot{\vee}$’s”** correspond to various *indeterminacies* that arise mainly from the **$\mathfrak{log}$-Kummer-correspondence**, i.e., from sequences of iterates of the **$\mathfrak{log}$-link** of inter-universal Teichmüller theory, which may be thought of as a device for constructing **additive** log-shells.

This sort of concatenation of *logical AND “$\wedge$’s”* and *logical XOR “$\dot{\vee}$’s”* is reminiscent of the well-known description of the **“carry-addition”** operation on Teichmüller representatives of the *truncated Witt ring* $\mathbb{Z}/4\mathbb{Z}$ in terms of **Boolean addition** “$\dot{\vee}$” and **Boolean multiplication** “$\wedge$” in the *field* $\mathbb{F}_2$ and may be regarded as a sort of **“Boolean intertwining”** that *mirrors*, in a *remarkable* fashion, the **“arithmetic intertwining”** between **addition** and **multiplication** in number fields and local fields, which is, in some sense, the main object of study in inter-universal Teichmüller theory. One important topic in this exposition is the issue of **“redundant copies”**, i.e., the issue of how the **arbitrary identification** of copies of *isomorphic mathematical objects* that appear in the various constructions of inter-universal Teichmüller theory impacts — and indeed *invalidates* — the *essential logical structure* of inter-universal Teichmüller theory. This issue has been a focal point of **fundamental misunderstandings** and **entirely unnecessary confusion** concerning inter-universal Teichmüller theory in certain sectors of the mathematical community. The exposition of the topic of *“redundant copies”* makes use of many interesting **elementary** examples from the **history of mathematics**.

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Introduction

In the present paper, we give a detailed exposition of the **essential logical structure** of inter-universal Teichmüller theory in terms of elementary **Boolean operators** such as *logical AND "$\wedge$"* and *logical OR "$\vee$"*. One important topic in this exposition is the issue of **"redundant copies"**, i.e., the issue of how the **arbitrary identification** of copies of *isomorphic mathematical objects* that appear in the various constructions of inter-universal Teichmüller theory impacts — and indeed *invalidates* — the *essential logical structure* of inter-universal Teichmüller theory. This issue has been a focal point of **fundamental misunderstandings** and **entirely unnecessary confusion** concerning inter-universal Teichmüller theory in certain sectors of the mathematical community [cf. the discussion of Examples 2.4.5, 2.4.7, 2.4.8].

We begin, in §1, by reporting on various *non-mathematical* aspects of the situation surrounding inter-universal Teichmüller theory, such as the issue of *"redundant copies"*. Perhaps the most central portion of this discussion of non-mathematical aspects of the situation surrounding inter-universal Teichmüller theory concerns the **long-term, historical importance** of producing *detailed, explicit, mathematically substantive*, and *readily accessible* **written documentation** of the **essential logical structure** of the issues under debate [cf. §1.5]. Such *written documentation* of the *essential logical structure* of the issues under debate is *especially important* in situations such as the situation that has arisen surrounding inter-universal Teichmüller theory, in which the proliferation of **logically unrelated fabricated versions** of the theory has led to *fundamental misunderstandings* and *entirely unnecessary confusion* concerning inter-universal Teichmüller theory in certain sectors of the mathematical community that are **deeply detrimental** to the **operational normalcy** of the field of mathematics [cf. §1.3, §1.8, §1.10, §1.11, §1.12]. This discussion in §1 is supplemented by various interesting **historical examples** related to the **irrationality** of square roots of prime numbers [cf. Example 1.5.1], the **Pythagorean Theorem** [cf. Example 1.10.1], and **Pythagorean triples** [cf. Example 1.12.1].

We then proceed, in §2, to discuss *elementary aspects* of the mathematics surrounding the essential logical structure of inter-universal Teichmüller theory. Our discussion of these elementary aspects, which concerns mathematics at the *advanced undergraduate* or *beginning graduate* level and does *not* require any advanced knowledge of *anabelian geometry* or *inter-universal Teichmüller theory*, focuses on the close relationship between

- **integration** and **differentiation** on — i.e., so to speak, the *"differential geometry"* of — the **real line** [cf. §2.1, §2.2, as well as Example 2.4.2],
- the *geometry* of **adjacent closed intervals** of the **real line** and the **loops** that arise by *identifying* various closed subspaces of such closed intervals [cf. §2.3; Example 2.4.1], and
- **Boolean operators** such as *logical AND "$\wedge$"* and *logical OR "$\vee$"* [cf. §2.4].

One *important unifying theme* that relates these seemingly disparate topics is the theme of **"carry operations"**, which appear in the various **arithmetic**, **geometric** [i.e., **"gluing"**], and **Boolean-logical** situations discussed in §2 [cf. Example 2.4.6].

On the other hand, from the point of view of *arithmetic geometry*, the discussion of

> the **projective line** as a **gluing** of **ring schemes** along a **multiplicative group scheme**

given in Example 2.4.7 yields a *remarkably elementary qualitative model/analogue* of the essential logical structure surrounding the gluing given by the **Θ-link** in inter-universal Teichmüller theory. Moreover, over the complex numbers, this example of the projective line — i.e., which may be visualized as a *sphere* — leads to an interesting analogy between the well-known [e.g., especially, in a *cartographic* context!] **metric/geodesic geometry** of the **sphere** with the **multiradial representation** of the **Θ-pilot** in inter-universal Teichmüller theory [cf. Example 2.4.7, (v)]. This example of the projective line discussed in Example 2.4.7 may be understood as occupying a *special role* in the exposition of the present paper in light of the fact that it is *more directly related to* **scheme-theoretic arithmetic geometry** than the previously mentioned examples and leads naturally to the subsequent **ring-/monoid-theoretic** Example 2.4.8, which may literally be regarded, i.e., in a much more *rigorous, technical sense*, as a sort of miniature qualitative model — that is to say, so to speak, a sort of *"preview"* — of the *gluing* constituted by the

Θ-link of inter-universal Teichmüller theory. Finally, this example of the projective line is also of interest in light of the *remarkable parallels* between the issue of **"redundant copies"** in the context of inter-universal Teichmüller theory and the well-known 19-th century **"algebraic truths"** versus **"geometric fantasies"** dispute between Weierstrass and Riemann concerning approaches to complex function theory [cf. the discussion of §1.5].

The preparatory topics of §2 lead naturally to the detailed exposition of the *essential logical structure* of inter-universal Teichmüller theory given in §3. From a strictly rigorous point of view, this exposition assumes a substantial level of knowledge and understanding of the technicalities of inter-universal Teichmüller theory [which are surveyed, for instance, in [Alien]], although the essential mathematical content of most of the issues discussed may in fact be understood at the level of the elementary considerations discussed in §2. The **essential logical structure** of inter-universal Teichmüller theory may be represented **symbolically** as follows:

$$\begin{array}{rcl} A \wedge B & = & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ \ldots) \\ & \Longrightarrow & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ \ldots \ \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \ \ldots) \\ & \Longrightarrow & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ \ldots \ \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \ \ldots \ \dot{\vee} B''_1 \dot{\vee} B''_2 \dot{\vee} \ \ldots) \\ & & \vdots \end{array}$$

— cf. the discussion of ($\wedge(\dot{\vee})$-Chn) in §3.10. [Here, "$\dot{\vee}$" denotes the Boolean operator *exclusive-OR*, i.e., *"XOR"*.] Indeed, §3 is devoted, for the most part, to giving a detailed exposition of various aspects of this *symbolic representation*, such as the following:

- the **logical AND "$\wedge$'s"** in the above display may be understood as corresponding to the **Θ-link** of inter-universal Teichmüller theory and are closely related to the **multiplicative** structures of the rings that appear in the domain and codomain of the Θ-link;
- the **logical XOR "$\dot{\vee}$'s"** in the above display may be understood as corresponding to various *indeterminacies* that arise mainly from the **$\mathfrak{log}$-Kummer-correspondence**, i.e., from sequences of iterates of the **$\mathfrak{log}$-link** of inter-universal Teichmüller theory, which may be thought of as a device for constructing **additive** log-shells.

This appearance of *logical AND "$\wedge$'s"* and *logical XOR "$\dot{\vee}$'s"* is of interest in that it is reminiscent of the well-known description of the **"carry-addition"** operation on Teichmüller representatives of the *truncated Witt ring* $\mathbb{Z}/4\mathbb{Z}$ in terms of **Boolean addition** "$\dot{\vee}$" and **Boolean multiplication** "$\wedge$" in the *field* $\mathbb{F}_2$ and may be regarded as a sort of **"Boolean intertwining"** that *mirrors*, in a *remarkable* fashion, the **"arithmetic intertwining"** between **addition** and **multiplication** in number fields and local fields, which is, in some sense, the main object of study in inter-universal Teichmüller theory [cf. the discussion of Example 2.4.6, (iii); the discussion surrounding (TrHrc) in §3.10]. The above *symbolic representation* of the *essential logical structure* of inter-universal Teichmüller theory arises naturally from considerations concerning such *key topics* as

- the **coricity/symmetry/commutativity** properties of the *log-theta-lattice* [cf. Example 3.2.2] and the closely related significance of working with both **Frobenius-like** and **étale-like** objects [cf. §3.2];

- the closely intertwined properties of **theta functions** and **[p-adic/archimedean] logarithms** [cf. Examples 3.3.2, 3.3.3, 3.8.3, 3.8.4] in the context of the **$\mathfrak{log}$-Kummer-correspondence** [cf. §3.3; §3.11; Examples 3.2.1, 3.3.1];
- generalities concerning **gluings** [cf. §3.1, §3.4, §3.5, §3.10; Examples 3.1.1, 3.5.2, 3.10.1, 3.10.2];
- generalities concerning **indeterminacies** [cf. §3.5, §3.6, §3.7; Example 3.5.1];
- generalities concerning **inter-universality** [cf. §3.8; Examples 3.8.1, 3.8.2, 3.8.3, 3.8.4];
- generalities concerning **descent to underlying structures** [cf. §3.9, §3.10, §3.11].

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Section 1: Summary of non-mathematical aspects for non-specialists

We begin with an overall summary of *non-mathematical aspects* of the situation surrounding [IUTchI-IV], which may be of interest to both *non-mathematicians* and *mathematicians*. We also refer to [FsADT], [FKvid], [FsDss], [FsPio] for a discussion of various aspects of this situation from slightly different points of view.

§1.1. Publication of [IUTchI-IV]

The four main papers [IUTchI-IV] on inter-universal Teichmüller theory (IUT) were accepted for publication in the Publications of the Research Institute for Mathematical Sciences (PRIMS) on February 5, 2020. This was announced at an online video news conference held at Kyoto University on April 3, 2020. The four papers were subsequently published in several special volumes of PRIMS, a leading international journal in the field of mathematics with a distinguished history dating back over half a century.

The refereeing for these Special Volumes was overseen by an Editorial Board for the Special Volumes chaired by Professors Masaki Kashiwara and Akio Tamagawa. [Needless to say, as the author of these four papers, I was completely excluded from the activities of this Editorial Board for the Special Volumes.] Professor Kashiwara, a professor emeritus at RIMS, Kyoto University, is a global leader in the fields of algebraic analysis and representation theory. Professor Tamagawa, currently a professor at RIMS, Kyoto University, is a leading pioneer in the field of anabelian geometry and related research in arithmetic geometry. Here, it should be noted that, to a substantial extent,

> **inter-universal Teichmüller theory** *arose as an extension/application — developed by the author in the highly mathematically stimulating environment at RIMS, Kyoto University, over the course of roughly two decades [i.e., 1992 - 2012] — of precisely the sort of* **anabelian geometry** *that was pioneered by Tamagawa.*

It is for this reason that PRIMS stood out among mathematics journals worldwide as the most appropriate — i.e., in the sense of being by far the most [and indeed perhaps the only truly] *technically qualified* — journal for the task of refereeing and publishing the four papers [IUTchI-IV] on inter-universal Teichmüller theory.

Both Professors Kashiwara and Tamagawa have an outstandingly high international reputation, built up over distinguished careers that span several decades. It is entirely inconceivable that any refereeing process overseen by these mathematicians might be conducted relative to anything less than the highest mathematical standards, free of any inappropriate non-mathematical considerations. In an article in the Asahi Shimbun [a major Japanese newspaper] published shortly after the announcement of April 3, 2020, Professor Tamagawa is quoted as saying that he has

"100 percent confidence in the refereeing"

that was done for the four papers [IUTchI-IV].

In another article in the Asahi Shimbun [also published shortly after the announcement of April 3, 2020], Professors Shigefumi Mori, a professor emeritus at RIMS, Kyoto University, and Nobushige Kurokawa, a professor emeritus at the Tokyo Institute of Technology, express their expectations about the possibility of applying inter-universal Teichmüller theory to other unsolved problems in number theory.

Subsequent to these developments in 2020, a sequel [ExpEst] to the four original papers [IUTchI-IV] on inter-universal Teichmüller theory was accepted for publication in the Kodai Mathematical Journal in September 2021. This sequel [ExpEst] concerns *explicit numerical estimates* in inter-universal Teichmüller theory and contains, in particular, a *new proof* of *Fermat's Last Theorem.*

In particular, the results proven in the four original papers [IUTchI-IV] on inter-universal Teichmüller theory, as well as the sequel [ExpEst], may now be quoted in the mathematical literature as results proven in papers that have been published in leading international journals in the field of mathematics after undergoing, in the case of the four original papers [IUTchI-IV], an exceptionally thorough [seven and a half year long] refereeing process.

§1.2. Redundancy assertions of the "redundant copies school" (RCS)

Unfortunately, it has been brought to my attention that, despite the developments discussed in §1.1, **fundamental misunderstandings** concerning the mathematical content of inter-universal Teichmüller theory persist in certain sectors of the mathematical community. These misunderstandings center around a certain *oversimplification* — which is *patently flawed*, i.e., leads to an immediate contradiction — of inter-universal Teichmüller theory. This oversimplified version of inter-universal Teichmüller theory is based on assertions of **redundancy** concerning various **multiple copies** of certain mathematical objects that appear in inter-universal Teichmüller theory. In the present paper, I shall refer to the *school of thought* [i.e., in the sense of a "collection of closely interrelated ideas"] constituted by these assertions as

the **"RCS"**, i.e., *"redundant copies school [of thought]".*

One fundamental reason for the use of this term "RCS" [i.e., "redundant copies school [of thought]"] in the present paper, as opposed to proper names of mathematicians, is to emphasize the importance of concentrating on **mathematical content**, as opposed to *non-mathematical* — i.e., such as *social*, *political*, or *psychological* — aspects or interpretations of the situation.

Thus, in a word, the central assertions of the RCS may be summarized as follows:

> Various **multiple copies** of certain mathematical objects in inter-universal Teichmüller theory are **redundant** and hence may be identified with one another. On the other hand, once one makes such **identifications**, one obtains an immediate **contradiction**.

In the present paper, I shall refer to redundancy in the sense of the assertions of the RCS as **"RCS-redundancy"**, to the identifications of RCS-redundant copies that appear in the assertions of the RCS as **"RCS-identifications"**, and to the oversimplified version of inter-universal Teichmüller theory obtained by implementing the RCS-identifications as **"RCS-IUT"**.

As discussed in [Rpt2018] [cf., especially, [Rpt2018], §18], there is *absolutely no doubt* that

> *RCS-IUT is indeed a meaningless and absurd theory that leads immediately to a contradiction.*

A more technical discussion of this contradiction, in the language of inter-universal Teichmüller theory, is given in §3.1 below, while digested versions in more elementary language of the technical discussion of §3 may be found in Examples 2.4.5, 2.4.7, 2.4.8, below.

Rather, the *fundamental misunderstandings* underlying the RCS lie in the assertions of *RCS-redundancy.* The usual sense of the word "redundant" suggests that there should be some sort of **equivalence**, or **close logical relationship**, between the original version of the theory [i.e., IUT] and the theory obtained [i.e., RCS-IUT] by implementing the RCS-identifications of RCS-redundant objects. In fact, however,

> implementing the RCS-identifications of RCS-redundant objects **radically alters/invalidates** the **essential logical structure** of IUT

in such a fundamental way that it seems entirely unrealistic to verify any sort of "close logical relationship" between IUT and RCS-IUT.

A more technical discussion of the *three main types* of *RCS-redundancy/RCS-identification* — which we refer to as "(RC-FrÉt)", "(RC-$\mathfrak{log}$)", and "(RC-Θ)" — is given, in the language of inter-universal Teichmüller theory, in §3.2, §3.3, §3.4, below. In fact, however, the **essential mathematical content** of these three main types of RCS-redundancy/RCS-identification is **entirely elementary** and lies well within the framework of *undergraduate-level mathematics.* A discussion of this essentially elementary mathematical content is given in §2.3, §2.4 below [cf., especially, Examples 2.4.5, 2.4.7, 2.4.8].

One important consequence of the technical considerations discussed in §3 below is the following:

> from the point of view of the **logical relationships** between various assertions of the RCS, the **most fundamental type** of RCS-redundancy is (RC-Θ).

That is to say, (RC-Θ) may be understood as the **logical cornerstone** of the various assertions of the RCS.

§**1.3.** **Qualitative assessment of assertions of the RCS**

As discussed in detail in §3.4 below [cf. also §2.3, §2.4],

> implementing the **logical cornerstone RCS-identification** of (RC-Θ) **completely invalidates** the crucial **logical AND "$\wedge$" property** satisfied by the Θ-link — a property that underlies the **entire logical structure** of inter-universal Teichmüller theory.

In particular, understanding the issue of how the RCS treats this *fundamental conflict* between the RCS-identification of (RC-Θ) and the crucial $\wedge$*-property* of the Θ-link is central to the issue of assessing the assertions of the RCS.

In March 2018, discussions were held at RIMS with two adherents of the RCS concerning, in particular, (RC-Θ) [cf. [Rpt2018], [Dsc2018]]. Subsequent to these discussions, after a few e-mail exchanges, these two adherents of the RCS informed me via e-mail in August 2018 — in response to an e-mail that I sent to them in which I stated that I was prepared to continue discussing inter-universal Teichmüller theory with them, but that I had gotten the impression that they were not interested in continuing these discussions — that indeed they were *not interested in continuing these discussions concerning inter-universal Teichmüller theory.* In the same e-mail, I also stated that perhaps it might be more productive to continue these discussions of inter-universal Teichmüller theory via *different participants* [i.e., via *"representatives"* of the two sides] and encouraged them to suggest *possible candidates* for doing this, but they never responded to this portion of my e-mail. [Incidentally, it should be understood that I have no objection to making these e-mail messages public, but will refrain from doing so in the absence of explicit permission from the two recipients of the e-mails.]

Since March 2018, I have spent a tremendous amount of time discussing the **fundamental "(RC-Θ) vs. $\wedge$-property" conflict** mentioned above with quite a number of mathematicians. Moreover, during the years following the March 2018 discussions, many mathematicians [including myself!] with whom I have been in contact have devoted a quite substantial amount of time and effort to analyzing and discussing certain 10pp. manuscripts written by adherents of the RCS [cf., especially, the discussion of the *final page and a half* of the files "[SS2018-05]", "[SS2018-08]" available at the website [Dsc2018]] — indeed to such an extent that by now, many of us can cite numerous key passages in these manuscripts by memory. More recently, one mathematician with whom I have been in contact has made a quite intensive study of the mathematical content of recent blog posts by adherents of the RCS.

Despite all of these efforts, the **only justification** for the **logical cornerstone RCS-identification** of (RC-Θ) that we [i.e., I myself, together with the many mathematicians with whom I have discussed these issues] could find either in oral explanations during the discussions of March 2018 or in subsequent written records produced by adherents of the RCS [i.e., such as the 10pp. manuscripts referred to above or various blog posts] were statements of the form

"I don't see why not".

[I continue to find it *utterly bizarre* that such justifications of the assertions of the RCS appear to be taken seriously by some professional mathematicians.] In particular, we were *unable to find any detailed mathematical discussion* by adherents of the RCS of the **fundamental "(RC-Θ) vs. $\wedge$-property" conflict** mentioned above. That is to say, in summary,

> the **mathematical justification** for the **"redundancy"** asserted in the logical cornerstone assertion (RC-Θ) of the RCS remains a **complete mystery** to myself, as well as to all of the mathematicians that I have consulted concerning this issue

[cf. the discussion of Examples 2.4.5, 2.4.7, 2.4.8]. Put another way, the response of all of the mathematicians with whom I have had technically meaningful discussions concerning the assertions of the RCS was *completely uniform* and *unanimous*, i.e., to the effect that these assertions of the RCS were **obviously completely mathematically inaccurate/absurd**, and that they had no idea why adherents of the RCS continued to make such manifestly absurd assertions. In particular, it should be emphasized that

> *I continue to search for a professional mathematician [say, in the field of arithmetic geometry] who feels that he/she understands the mathematical content of the assertions of the RCS and is willing to discuss this mathematical content with me or other mathematicians with whom I am in contact*

[cf. the text at the beginning of [Dsc2018]]. It is worth noting that this situation also constitutes a **serious violation** of article (6.)

> *Mathematicians should not make public claims of potential new theorems or the resolution of particular mathematical problems unless they are able to provide full details in a timely manner.*

of the subsection entitled "Responsibilities of authors" of the Code of Practice of the European Mathematical Society (cf. [EMSCOP]).

In this context, one important observation that should be kept in mind is the following [cf. the discussion of [Rpt2018], §18]:

(UndIg) There is a *fundamental difference* between

(UndIg1) criticism of a mathematical theory that is based on a **solid, technically accurate understanding** of the content and logical structure of the theory and

(UndIg2) criticism of a mathematical theory that is based on a **fundamental ignorance** of the content and logical structure of the theory.

An elementary classical example of this sort of difference is discussed in §2.1 below.

In the case of the RCS, the lack of any thorough mathematical discussion of the **fundamental "(RC-Θ) vs. $\wedge$-property" conflict** mentioned above in the various oral/written explanations set forth by adherents of the RCS demonstrates, in a definitive way, that *none of the adherents of the RCS has a solid, technically accurate understanding of the logical structure of inter-universal Teichmüller theory in its original form*, i.e., in particular, of the central role played in this logical structure by the "$\wedge$-property" of the Θ-link. Put another way, the only logically consistent explanation of this state of affairs is that the theory "RCS-IUT" that adherents of the RCS have in mind, i.e., the theory that is the object of their criticism, is simply a **completely different** — and **logically unrelated** — **theory** from the theory constituted by inter-universal Teichmüller theory in its original form.

Finally, it should be mentioned that although some people have asserted parallels between the assertions of the RCS and the *fundamental error in the first version of Wiles's proof of the Modularity Conjecture in the mid-1990's*, this analogy is **entirely inappropriate** for numerous reasons. Indeed, as is well-known, nothing even remotely close to the phenomena discussed thus far in the present §1.3 occurred in the case of the error in the first version of Wiles's proof. The fact that there was indeed a fatal error in the first version of Wiles's proof was never disputed in any way by any of the parties involved; the only issue that arose was the issue of whether or not the proof could be fixed. By contrast, *no essential errors* have been found in inter-universal Teichmüller theory, since the four preprints [IUTchI-IV] on inter-universal Teichmüller theory were released in August 2012. That is to say, in a word, the assertions of the RCS are nothing more than **meaningless, superficial misunderstandings** of inter-universal Teichmüller theory on the part of people who are clearly *not operating on the basis of a solid, technically accurate understanding of the mathematical content and essential logical structure* of inter-universal Teichmüller theory.

§1.4. The importance of extensive, long-term interaction

In general, the **transmission of mathematical ideas** between individuals who share a sufficient stock of **common mathematical culture** may be achieved in a relatively efficient way and in a relatively brief amount of time. Typical examples of this sort of situation in the context of interaction between professional mathematicians include

- *one-hour mathematical lectures*,
- *week-long mathematical lecture series*, and
- *informal mathematical discussions for several days to a week.*

In the context of mathematical education, typical examples include

- *written or oral mathematical examinations* and
- *mathematics competitions.*

The successful operation of each of these examples relies, in an essential way, on a **common framework of mathematical culture** that is shared by the various participants in the activity under consideration.

On the other hand, in the case of a **fundamentally new** area of research, such as inter-universal Teichmüller theory, which evolved out of research over the past quarter of a century concerning *absolute anabelian geometry*, certain types of *categories arising from arithmetic geometry*, and certain arithmetic aspects of *theta functions*, the collection of mathematicians who share such a sufficient stock of common mathematical culture tends to be relatively small in number. In particular, for most mathematicians — even many arithmetic geometers or anabelian geometers — short-term interaction of the sort that occurs in the various typical examples mentioned above is **far from sufficient** to achieve an effective transmission of mathematical ideas. That is to say, no matter how mathematically talented the participants in such platforms of interaction may be, it takes time for the participants to

- *analyze and sort out numerous mutual misunderstandings*,
- *develop effective techniques of communication* that can transcend such misunderstandings, and
- *digest and absorb new ideas and modes of thought.*

Depending on the mathematical content under consideration, as well as on the mathematical talent, mathematical background, and time constraints of the participants, this painstaking process of analysis/development/digestion/absorption may require

> *patiently sustained efforts to continue constructive, orderly mathematical discussions [via e-mail, online video discussions, or face-to-face meetings] over a period of* **months** *or even* **years**

to reach fruition. Indeed, my experience in exposing the ideas of inter-universal Teichmüller theory to numerous mathematicians over the past decade suggests strongly that, in the case of inter-universal Teichmüller theory, it is difficult to expedite this process to the extent that it can be satisfactorily achieved in less than half a year or so.

In particular, in the case of inter-universal Teichmüller theory, a week-long session of discussions such as the discussions held at RIMS in March 2018 with two adherents of the RCS [cf. [Rpt2018], [Dsc2018]] is *far from sufficient.* This is something that I emphasized, both orally during these discussions and in e-mails to these two adherents of the RCS during the summer of 2018 subsequent to these discussions.

§1.5. The historical significance of detailed, explicit, accessible records

As was discussed in §1.3, I *continue to search* for a professional mathematician [say, in the field of arithmetic geometry] who purports to understand the **mathematical justification** for the **RCS-redundancy** asserted in the logical cornerstone assertion (RC-Θ) — i.e., in particular, who has confronted the mathematical content of the **fundamental "(RC-Θ) vs. $\wedge$-property" conflict** mentioned in §1.3 — and who is prepared to discuss this mathematical content with me or other mathematicians with whom I am in contact. Of course,

> a **detailed, explicit, mathematically substantive**, and **readily accessible written exposition**

— i.e., as an alternative to direct mathematical discussions [via e-mail, online video discussions, or face-to-face meetings] — of the *mathematical justification* for the logical cornerstone assertion (RC-Θ) would also be quite welcome [cf. the discussion of [Rpt2014], (7)]. Moreover, in this context, it should be *emphasized* that such a detailed, explicit, mathematically substantive, and readily accessible written exposition would be of great value not only for *professional mathematicians and graduate students* who are involved with inter-universal Teichmüller theory at the **present time**, but also for scholars in the **[perhaps distant!] future**.

In general, it cannot be overemphasized that maintaining such detailed, explicit, mathematically substantive, and readily accessible written records is

> of **fundamental importance** to the **development** of **mathematics**.

Indeed, as was discussed in the final portion of [Rpt2018], §3, from a **historical** point of view, it is only by maintaining such written records that the field of mathematics can avoid the sort of well-known and well-documented **confusion** that lasted for so many centuries concerning *"Fermat's Last Theorem"*. Moreover, it is fascinating to re-examine, from the point of view of a modern observer, the *intense debates* that occurred, during the time of *Galileo*, concerning the *theory of heliocentrism* or, during the time of *Einstein*, concerning the *theory of relativity*.

Perhaps a more pertinent example [as was pointed out to the author by *Fumiharu Kato*], relative to the issue of **"redundant copies"**, may be found in the well-known dispute — i.e., concerning *"algebraic truths"* [which corresponds to Weierstrass' approach] versus *"geometric fantasies"* [which corresponds to Riemann's approach] — in the 19-th century between *Weierstrass* and *Riemann* concerning approaches to complex function theory [cf. [Btt]]. That is to say, the criticism by Weierstrass of the "geometric fantasies" approach due to Riemann via **analytic continuation** on **Riemann surfaces**, which are obtained by **gluing** together — what are, perhaps to some observers, seemingly **"redundant"**! — **copies** of open subsets of the complex plane [cf. the discussion of the well-known gluing construction of the *projective line* in Example 2.4.7 below; the discussion of *analytic continuation* of the *complex logarithm* in the discussion surrounding (FxEuc) in §3.1 below, as well as in Example 3.10.1 below; the illustrations of [AnCnCv], [AnCnLg]!], exhibits **remarkable parallels** in spirit to the assertions of the RCS.

Here, it is also of interest to note [as was pointed out to the author by *Benjamin Collas*] that the issue of *analytic continuation* of the *complex logarithm* is very closely related to the long and heated dispute between *Leibniz* and *[Johann]*

Bernoulli concerning the appropriate definition of logarithms of *negative* and more general *complex numbers* — a dispute that was ultimately, in some sense, resolved by *Euler's formula* and the acceptance of the *multi-valued nature* of the *complex logarithm* [cf. the discussion of *"Euler's Formula and Its Consequences"* in [AnHst], Chapter I, §I.5, although it must be kept in mind that in Euler's time, complex function theory, i.e., of the sort necessary to treat the complex logarithm as a *holomorphic function*, had not yet been developed]. This **multi-valued nature** of the **complex logarithm** is closely related not only to the theory of **analytic continuation** of the *complex logarithm*, but also to the theory of *coverings* [in the sense of the *classical topological fundamental group*], where the indeterminacy in *values* may be understood as an indeterminacy in the choice of **basepoint**, that is to say, as a sort of **distant** — though nonetheless quite **direct**! — **ancestor** of the **inter-universal indeterminacies** that appear in inter-universal Teichmüller theory [cf. the discussion of §3.8, especially Example 3.8.1, below!].

The central role occupied by the notion of *analytic continuation* in these historical disputes of Weierstrass/Riemann and Leibniz/Bernoulli is also of interest, in the context of inter-universal Teichmüller theory, in light of the central role played, in the history of *analytic continuation*, by the **functional equation** [i.e., **Jacobi identity**] for the **theta function**, which exhibits *numerous remarkable structural similarities* to inter-universal Teichmüller theory [cf. the discussion of Example 3.3.2 below], and which is also closely related to Riemann's famous research concerning the theory of the *functional equation/analytic continuation* of the **Riemann zeta function**. Research in the 19-th century concerning **analytic continuation** may also be regarded as a sort of early **precursor** of more modern notions such as **monodromy representations** and **connections/crystals** — an observation which is of interest in light of the analogy between *connections/crystals* and the notion of *multiradiality* in inter-universal Teichmüller theory [cf. the discussion of [Alien], §3.1, (v), as well as the discussion of §3.5, §3.10 below].

Before proceeding, we remark that it is interesting to observe that this historical discussion of the *functional equation* of the *theta function* in the context of *analytic continuation*, in this case on the *upper half-plane*, is reminiscent of the *famous observation* of *Poincaré* — i.e., in the form of an idea that came to him while in transit during his travels — that "the transformations" that he had used "to define Fuchsian functions were identical with those of non-Euclidian geometry" [cf. [Pnc], p. 53]. This famous observation concerning the *isomorphic* nature of the *group of transformations* of a modular function — i.e., such as the *theta function* — and a certain *group of symmetries* of the ["non-Euclidean"] *hyperbolic geometry* of the *upper half-plane* seems to be one of the principal motivations behind the *famous quote*, due to *Poincaré*, that

> *"mathematics is the art of giving the same name to different things"*, i.e.,
> *"things which differ in matter, but are similar in form"*

[cf. the discussion of [Pnc], pp. 34 – 35]. Moreover, this *remarkable train of thought*, which was recorded for posterity by Poincaré in [Pnc], is of particular interest in the context of the present discussion of the relationship between various classical notions and inter-universal Teichmüller theory in that it seems almost *prescient* in its *deep resemblance* to the main notions — namely, *coricity* and *multiradiality*

— that underlie the concept of **inter-universality** in inter-universal Teichmüller theory. Indeed:

(SmDff1) The search for **coric** structures in inter-universal Teichmüller theory may be thought of precisely as the search for the *"same name"* for suitable portions of structures on opposite sides — i.e., *"different things"* — of the Θ- or $\mathfrak{log}$-links in inter-universal Teichmüller theory. Moreover, perhaps the most fundamental instance of such a coric structure in inter-universal Teichmüller theory consists of the *abstract topological groups*, regarded up to *isomorphism*, underlying the Galois groups/arithmetic fundamental groups that arise from rings/schemes on opposite sides of the Θ- or $\mathfrak{log}$-links.

(SmDff2) The search for **multiradial** structures in inter-universal Teichmüller theory may be thought of precisely as the search for the *"same name"* — i.e., more precisely, in the form of an *isomorphism up to suitable indeterminacies* — for

- a suitable portion of the *system* [or portion of the *log-theta-lattice*] consisting of *distinct* ring/scheme structures linked by the Θ-*link*, on the one hand, and
- a suitable portion of a *single* ring/scheme structure, on the other

— i.e., for *"things which differ in matter"*.

More technical details may be found in the discussion of *coricity* and *inter-universality* in §3.2, §3.8, below, as well in the discussion of *descent* via the *multiradial algorithms* of inter-universal Teichmüller theory given in §3.10, §3.11, below.

Of course, from the point of view of a modern observer who is well-versed in **axiomatic set theory** and the theory of **[Grothendieck] universes**, the bitter historical disputes of Weierstrass/Riemann and Leibniz/Bernoulli discussed above seem somewhat *quaint* or even *"silly"*. One should not, however, in this context, overlook the importance of such bitter historical disputes in motivating the development of such modern tools as *axiomatic set theory* and the theory of *[Grothendieck] universes*, which *underlie this privileged viewpoint of the modern observer*. Conversely, when viewed from this *historical perspective*, the assertions of the RCS seem all the more like a sort of **bizarre anachronism**, which has *no place in the 21-st century* [cf., especially, the *entirely elementary* nature of the various examples — such as the gluing construction of the projective line discussed in Example 2.4.7 — that appear in §2 below]! Moreover, in the case of the Leibniz/Bernoulli dispute, it is of interest to note — especially, in the context of the **highly sensationalist** nature of the coverage of inter-universal Teichmüller theory in the *English-language mass-media* and *internet* [cf., e.g., the discussion of §1.8, §1.10, §1.12 below] — that apparently, at the time of the dispute, the dispute was kept as *secret as possible* in order to avoid *"damaging the prestige of pure mathematics as an exact and rigorous science"* [cf. the discussion of *"Euler's Formula and Its Consequences"* in [AnHst], Chapter I, §I.5]. At any rate, in the context of this sort of historical discussion, it cannot be overemphasized that

> all of these *historical re-examinations* [i.e., of the sort that underlie the discussion of the last few paragraphs!] are technically possible precisely

because of the existence of **detailed, explicit, mathematically substantive**, and **readily accessible written expositions** of the **logical structure** underlying the various central assertions that arose in the debate.

In this context, it should be noted that, from a historical point of view, one pattern that typically underlies the **formidable deadlocks** that tend to occur in such debates is the point of view, on the part of parties opposed to a newly developed theory, that

(CmSn) it is a **"matter of course"** or **"common sense"** — i.e., in the language of the above discussion, a matter that is so *profoundly self-evident* that any *"decent, reasonable observer"* would undoubtedly find *detailed, explicit, mathematically substantive, and readily accessible* **written expositions** of its **logical structure** to be **entirely unnecessary** — that the issues under consideration can be completely resolved within some existing, familiar framework of thought without the introduction of the newly developed theory, which is regarded as *deeply disturbing* and *unlikely to be of use in any substantive mathematical sense*

— cf., e.g., the discussion of Example 1.5.1 below. In fact, however,

(OvDlk) ultimately, the *only meaningful technical tool* that humanity can apply to develop the **cultural infrastructure** necessary to **overcome such deadlocks** is precisely the production of *detailed, explicit, mathematically substantive, and readily accessible* **written expositions** of the **logical structure** underlying the points of view that are regarded by certain parties as being a *"matter of common sense"*

[cf. the discussion of [EMSCOP] in §1.3; the discussion of (UndIg) in §1.3; the discussion of §1.10 below; §2.1 below; the discussion of (FxRng), (FxEuc), (FxFld), (RdVar) in §3.1 below].

Example 1.5.1: Irrationality, impartiality, and the Voodoo Hypothesis.

(i) We begin by recalling the very classical and elementary proof that, for any *prime number* p, $\sqrt{p}$ is **irrational**, i.e., that there does not exist any rational number whose square is equal to p. Indeed, if there did exist a rational number $r = a/b$, where a and b are relatively prime nonzero integers, such that $r^2 = p$, then the resulting relation $a^2 = p \cdot b^2$ would violate the *unique factorization property* satisfied by the nonzero integers. This unique factorization property may, in turn, by verified by applying the *Euclidean division algorithm.*

(ii) The discovery of the irrationality of square roots of prime numbers discussed in (i) is typically attributed, in the case $p = 2$ [in which case the Euclidean division algorithm amounts, in essence, to the classification of integers into *odd* and *even* integers], to the ancient Greek philosopher *Hippasus.* Apparently, this discovery arose in the context of applying the *Pythagorean Theorem* [cf. the discussion of Examples 1.10.1, 1.12.1 below] to compute the length of the diagonal of a square with sides of length 1. The *Pythagorean school* was reported to have found this discovery to be *shocking* and *deeply disturbing*, and indeed it seems that this

negative appraisal of Hippasus' discovery may have led to the death of Hippasus by drowning.

(iii) The sort of *negative appraisal* that occurred in (ii) could easily arise in the mind of any observer — indeed, *even modern observers* [such as students] who are not familiar with the sort of *abstract mathematical reasoning* that underlies the proof of (i) — who has a deep sense of confidence in his/her understanding of the *"common sense definition of mathematics"* as the study of explicit computations involving rational numbers, e.g., of the sort that may be done with a desktop calculator [or, in earlier centuries, with an abacus!]. [We refer to Example 1.5.2 below for further similar examples of this sort of phenomenon.] Such a negative appraisal on the part of a strongly *computationally oriented observer* might also be additionally supported by

(HeurBlf) a *heuristically supported* **belief** that "surely" if one is given sufficient time and manpower to perform suitable computations to *sufficiently high order*, then there is no doubt that one should be able to find a *rational square root of* 2.

Moreover, depending on the social/political circumstances surrounding the situation, even third parties who are ignorant of the details of situation might be led

(FsObjImpl) to [*falsely*!] assert that the *only truly* **objective** *or* **impartial** way to treat the two schools — i.e., of people who assert the *irrationality* of $\sqrt{2}$ via the proof of (i) and people who doubt this argument — is to talk as if the issue of the irrationality of $\sqrt{2}$ is *unresolvable* or even to *refuse to discuss* the issue at all.

Indeed, it is worth recalling that ancient accounts of Hippasus and the Pythagorean school are a *stark reminder* of just how dire the consequences can be when those who are *socially/politically* recognized as *"objective arbiters"* for a mathematical dispute act on the basis of a *grossly mathematically inaccurate* understanding of the situation.

(iv) In the context of the discussion of (iii), it is useful to refer to a situation [i.e., such as the situation discussed in (iii)] in which *the validity of a mathematical proof is called into question*, **not** on the basis of some sort of logical defect [i.e., such as a gap in the proof], but on the basis of some sort of **heuristically based belief** [cf. (HeurBlf)] to the effect that "surely" if one is given sufficient time and manpower to sort through various *technical details*, then there is no doubt that one should be able to find some sort of substantive problem with the proof — such as a *counterexample* or *fallacious reasoning*, i.e., some sort of *"voodoo"* — as an invocation of the **"Voodoo Hypothesis"**. Once the *Voodoo Hypothesis* has been invoked, mathematicians who have a mathematically accurate, rigorous understanding of the proof in question are then often portrayed as being nothing more than **mindlessly obedient zombies**, i.e., who are acting solely or essentially under the influence of the "voodoo" applied in the proof. Indeed, this is precisely the sort of situation that has developed, in certain sectors of the mathematical community, concerning inter-universal Teichmüller theory — cf. the discussion of RCS-IUT in §1.2, §1.3, as well as §1.8, §1.10 below; Examples 2.4.5, 2.4.7, 2.4.8 below; the discussion of (FxRng), (FxEuc), (FxFld), (RdVar) in §3.1 below.

(v) At this point, we observe that the discussion of (ii), (iii), and (iv) prompt the following question:

> What *lessons* may be learned from the discussion in (ii), (iii), and (iv) of the *negative appraisal* of the proof of the irrationality of $\sqrt{2}$?

First of all, it is important to remember that

(Blf≠Pf) *heuristically based* **beliefs**, as in (HeurBlf), are **not mathematical proofs** and, in particular, can *never* serve as a *"viable substitute"* for a rigorous mathematical proof.

Secondly, it is important to remember [cf. (FsObjImpl)] that

(MthAcc) a truly **objective/impartial** position concerning a mathematical dispute can only be achieved as a result of a **mathematically accurate** understanding of the mathematics involved and, in particular, can *never* be achieved through an *ignorance* of the mathematics involved.

In this context, it is interesting to note that (MthAcc) has important consequences from the point of view of the topic of **computer verification** of mathematical assertions [cf. the discussion surrounding (CmpVer) in §1.12 below]. That is to say, for instance, in the case of the proof of (i), although it is not so technically difficult to formalize this [relatively simple!] argument in such a way that the argument could be "verified" by a computer,

(SocPol) such a computer verification becomes **entirely socially/politically meaningless** in situations in which parties — such as the *computationally oriented observer* of (iii)! — who do not share or recognize the *abstract conceptual mathematical framework* that necessarily underlies any sort of computer-ready formalization of the proof of (i) hold a monopoly on social/political authority.

Thus, in summary, from a *historical point of view*, it seems that the *main lesson* to be learned from the situation discussed in (ii), (iii), and (iv) is

(LTInfr) the *fundamental importance* — in the context of dealing in a meaningful and effective manner with mathematical disputes — of maintaining a **long-term infrastructural apparatus** for directly confronting, disseminating, and further developing the mathematics involved [cf. the discussion surrounding (BlkAcc) in §1.12 below].

Needless to say, the starting point of the activities of such a long-term infrastructural apparatus necessarily lies in the production of *detailed, explicit, mathematically substantive, and readily accessible* **written expositions** of the **logical structure** underlying the points of view that are regarded by various parties as being a *"matter of common sense"*.

Example 1.5.2: The internet/mass media as an apple of discord. In Example 1.5.1 [cf., especially, the discussion of *"modern observers"* in Example 1.5.1, (iii)], the particular example of the issue of the irrationality of $\sqrt{2}$ was examined in detail. In fact, however, this sort of situation exists in quite substantial abundance throughout mathematics and especially throughout the sort of elementary mathematics that is commonly covered in primary and secondary, as well as in university, eduation. Well-known examples of this phenomenon include the following:

(NegInt) the sense, in the context of multiplication of positive and negative integers, that *any product of two negative integers* must be a *negative* integer, i.e., on the grounds that the appearance of "two minus signs" in the product "surely" results in output that is *"all the more negative!"*;

(MxDiag) the sense, in the context of elementary linear algebra, that "surely" *all matrices are diagonalizable*, i.e., if one just tries hard enough to find the *"right basis!"*.

Moreover, unlike the case with the ancient context discussed in Example 1.5.1, (ii), this sort of phenomenon can be substantially further exacerbated in modern contexts by the *internet/mass media*, which exhibits a conspicuous tendency to operate as a sort of **"apple of discord"** that has the effect of not only creating, but also amplifying to often absurd proportions, an *artificial socio-political dynamic* that fuels a burning desire to *leap to conclusions* and achieve *"instant satisfaction"* with respect to some sort of essentially meaningless fictional/delusional mirage, e.g., with respect to achieving an understanding of some sort of apparently puzzling mathematical phenomenon of the sort described in (NegInt), (MxDiag) that appears, to some observers, to defy their *deep heuristic "common sense" understanding* of the situation — cf. the discussion of §1.8, §1.10, §1.12, below. Finally, we observe that, just as in the case of the discussion surrounding (LTInfr) in Example 1.5.1, (v), the only way, to the knowledge of the author at the time of writing, to overcome the detrimental effects of such artificial socio-political dynamics lies in maintaining a *long-term infrastructural apparatus* for directly confronting, disseminating, and further developing the mathematics involved, the starting point of which inevitably involves the production of *detailed, explicit, mathematically substantive, and readily accessible* **written expositions** of the **logical structure** underlying the points of view that are regarded by various parties as being a *"matter of common sense"*.

§**1.6.** **The importance of further dissemination**

One fundamental and frequently discussed theme in the further development of inter-universal Teichmüller theory is the issue of increasing the number of professional mathematicians who have a **solid, technically accurate understanding** of the details of inter-universal Teichmüller theory. Indeed, this issue is in some sense the central topic of [Rpt2013], [Rpt2014]. As discussed in §1.4, in order to achieve such a solid, technically accurate understanding of the theory, it is necessary to devote a substantial amount of *time* and *effort* over a period of roughly half a year to two or three years, depending on various factors. It also requires the participation of professional mathematicians or graduate students who are

- *sufficiently familiar with numerous more classical theories in arithmetic geometry* [cf. the discussion of [Alien], §4.1, (ii); [Alien], §4.4, (ii)],
- *sufficiently well motivated and enthusiastic* about studying inter-universal Teichmüller theory, and
- *sufficiently mathematically talented*, and who have a
- *sufficient amount of time* to devote to studying the theory.

As a result of quite substantial dissemination efforts not only on my part, but also on the part of many other mathematicians, the number of professional mathematicians who have achieved a sufficiently detailed understanding of inter-universal

Teichmüller theory to make independent, well informed, definitive statements concerning the theory that may be confirmed by existing experts on the theory [cf. also the discussion of §1.7 below] is roughly on the order of 10. It is worth noting that although this collection of mathematicians is centered around RIMS, Kyoto University, it includes mathematicians of many nationalities and of age ranging from around 30 to around 60. One recent example demonstrated quite dramatically that it is quite possible to achieve a solid mathematical understanding of inter-universal Teichmüller theory as a graduate student by studying on one's own, outside of Japan, and with essentially zero contact with RIMS, except for a very brief period of a few months at the final stage of the student's study of inter-universal Teichmüller theory.

Finally, we observe, in the context of the discussion [cf. §1.3, §1.4, §1.5] of the assertions of the RCS, that another point that should be emphasized is that it is also of fundamental importance to

> *increase the number of professional mathematicians [say, in the field of arithmetic geometry] who have a* **solid technical understanding** *of the mathematical content of the* **assertions** *of the* **RCS***, and who are prepared to discuss this mathematical content with members of the* **"IUT community"**

[i.e., with mathematicians who are substantially involved in mathematical research and/or dissemination activities concerning inter-universal Teichmüller theory]. Here, we note in passing that such a *solid technical understanding* of the mathematical content of the assertions of the RCS is *by no means "equivalent"* to expressions of support for the RCS on the basis of *non-mathematical* — i.e., such as *social*, *political*, or *psychological* — reasons. In this context, it should also be emphasized and understood [cf. the discussion of [Rpt2014], (7)] that *both*

- producing *detailed, explicit, mathematically substantive*, and *readily accessible written expositions* of the *mathematical justification* of assertions of the RCS [such as (RC-Θ)!], i.e., as discussed in §1.5, *and*
- increasing the number of professional mathematicians [say, in the field of arithmetic geometry] who have a *solid technical understanding* of the mathematical content of the assertions of the RCS, and who are prepared to discuss this mathematical content with members of the IUT community

are in the **interest** *not only* of the IUT community, but of the RCS *as well*. Moreover,

> the process of attaining a *solid, technically accurate understanding* of the **precise logical relationship** between RCS-IUT and IUT, i.e., as exposed, for instance, in the present paper, can serve as a **valuable pedagogical tool**

[cf. the discussion of [Rpt2018], §17] for mathematicians currently in the process of studying inter-universal Teichmüller theory.

§1.7. The notion of an "expert"

One topic that sometimes arises in the context of discussions of *dissemination* of inter-universal Teichmüller theory [i.e., as in §1.6], is the following issue:

> What is the *definition* of, or *criterion* for, being an **"expert"** on inter-universal Teichmüller theory?

In a word, it is very difficult to give a brief, definitive answer, e.g., in the form of a straightforward, easily applicable criterion, to this question. On the other hand, in this context, it should also be pointed out that the difficulties that arise in the case of inter-universal Teichmüller theory are, in fact, not so qualitatively different from the difficulties that arise in answering the *analogous question* for *mathematical theories other than inter-universal Teichmüller theory.* These difficulties arise throughout the daily life of professional mathematicians in numerous contexts, such as the following:

(Ev1) preparing suitable *exercises or examination problems to educate and evaluate students*,
(Ev2) *evaluating junior mathematicians*,
(Ev3) *refereeing/evaluating mathematical papers* for journals.

From my point of view, as the author of [IUTchI-IV], one *fundamental criterion* that I always keep in mind — not only the in case of [IUTchI-IV], but also in the case of other papers that I have written, as well as when I am involved in the various types of *evaluation procedures* (Ev1) $\sim$ (Ev3) discussed above — is the issue of

> *the extent to which the level of understanding of the mathematician in question enables the mathematician to "stand on his/her own two feet" with regard to various assertions concerning the theory, on the basis of independent, logical reasoning, without needing to be "propped up" or corrected by me or other known experts in the theory.*

I often refer to this criterion as the criterion of **autonomy** of **understanding**. Of course, from a strictly rigorous point of view, this criterion is, in some sense, not so "well-defined" and, in many contexts, difficult to apply in a straightforward fashion. On the other hand, in the past, various mathematicians involved with inter-universal Teichmüller theory have demonstrated such an *autonomous level of understanding* in the following ways:

(Atm1) the ability to detect various *minor errors/oversights* in [IUTchI-III];

(Atm2) the ability to propose *new, insightful ways of thinking* about various aspects of inter-universal Teichmüller theory;

(Atm3) the ability to propose *ways of modifying inter-universal Teichmüller theory* so as to yield *stronger or more efficient* versions of the theory;

(Atm4) the ability to produce *technically accurate oral or written expositions* of inter-universal Teichmüller theory;

(Atm5) the ability to supervise or direct *new mathematicians* — i.e., by training/educating professional mathematicians or graduate students with regard to inter-universal Teichmüller theory — who, in due time, demonstrate various of the four types of ability (Atm1) $\sim$ (Atm4) discussed above.

Of course, just as in the case of other mathematical theories, different experts demonstrate their expertise in different ways. That is to say, experts in inter-universal Teichmüller theory often demonstrate their expertise with respect to some of these five types of ability (Atm1) $\sim$ (Atm5), but not others.

In this context, it should be pointed out that one aspect of inter-universal Teichmüller theory that is currently still under development is the analogue for inter-universal Teichmüller theory of (Ev1), i.e., preparing suitable *exercises* for mathematicians currently in the process of studying inter-universal Teichmüller theory. This point of view may be seen in the discussion in the final portion of the Introduction to [Alien], as well as in the discussion of *"valuable pedagogical tools"* in [Rpt2018], §17 [cf. also the discussion in the final portion of §1.6 of the present paper]. Indeed, many of the technical issues discussed in [Rpt2018], §15 [or, alternatively, Example 3.2.2 of the present paper], may easily be reformulated as *"exercises"* or, alternatively, as *"examination problems"* for evaluating the level of understanding of mathematicians in the process of studying inter-universal Teichmüller theory.

§**1.8.** <u>**Fabricated versions spawn fabricated dramas**</u>

As discussed in §1.6, §1.7, by now there is a substantial number of mathematicians who have attained a *thorough, accurate, and automous understanding* of inter-universal Teichmüller theory. In each of the cases of such mathematicians that I have observed thus far, such an understanding of the theory was achieved essentially by means of a *thorough study of the original papers [IUTchI-IV]*, followed by a period of *constructive discussions with one or more existing experts* that typically lasted roughly from two to six months to sort out and resolve various "bugs" in the mathematician's understanding of the theory that arose when the mathematician studied the original papers on his/her own [cf. the discussion of §1.4].

On the other hand, there is also a growing collection of mathematicians who have a somewhat *inaccurate* and *incomplete* — and indeed often quite *superficial* — *understanding* of certain aspects of the theory. This in and of itself is not problematic — that is to say, so long as the mathematician in question maintains an *appropriate level of self-awareness* of the inaccurate and incomplete nature of his/her level of understanding of the theory — and indeed is a phenomenon that often occurs as abstract mathematical theories are disseminated.

Unfortunately, however, a certain portion of this collection of mathematicians [i.e., whose understanding of the theory is inaccurate and incomplete] have exhibited a tendency to

> assert/justify the validity of their inaccurate and incomplete understanding of the theory by means of **"reformulations"** or **"simplifications"** of the theory, which are in fact *substantively different from* and have *no directly logical relationship* to [e.g., are *by no means "equivalent"* to!] the original theory.

Indeed, the version, referred to in the present paper as "RCS-IUT" [cf. §1.2], that arises from implementing the assertions of the RCS appears to be the most famous of these fabricated versions of inter-universal Teichmüller theory [cf. also the discussion of Example 2.4.5 below for a more detailed discussion of various closely related variants of RCS-IUT]. On the other hand, other, less famous fabricated versions of inter-universal Teichmüller theory have also come to my attention in recent years.

Here, before proceeding, we note that, in general, reformulations or simplifications of a mathematical theory are *not necessarily problematic*, i.e., so long as they are indeed based on a *thorough and accurate understanding* of the original theory and, moreover, can be shown to have a *direct logical relationship* to the original theory.

The authoring of fabricated versions of inter-universal Teichmüller theory appears to be motivated, to a substantial extent, by a deep desire to recast inter-universal Teichmüller theory in a *"simplified"* form that is *much closer* to the sort of mathematics with which the author of the fabricated version *is already familiar/feels comfortable.* On the other hand, this phenomenon of producing fabricated versions also appears to have been

> substantially fueled by numerous *grotesquely distorted mass media reports* and *comments* on the English-language internet that blithely paint inter-universal Teichmüller theory as a sort of *cult religion*, *fanatical political movement*, *mystical philosophy*, or *vague sketch/proposal for a mathematical theory.*

Moreover, another unfortunate tendency, of which RCS-IUT is perhaps the most egregious example, is for fabricated versions of inter-universal Teichmüller theory to

> *spawn* **lurid social/political dramas** *revolving around the content of the fabricated version, which in fact have essentially nothing to do with the content of inter-universal Teichmüller theory.*

Such lurid dramas then spawn further grotesquely distorted mass media reports and comments on the English-language internet, which then reinforce and enhance the social/political status of the fabricated version [cf. the discussion of Example 1.5.2]. Here, it should be emphasized that such **vicious spirals** have little [or nothing] to do with substantive mathematical content and indeed serve only to mass-produce **unnecessary confusion** that is entirely counterproductive, from the point of the view of charting a sound, sustainable course in the future development of the field of mathematics [cf. the discussion of [Alien], §4.4, (iv)].

In fact, of course, inter-universal Teichmüller theory is neither a religion, nor a political movement, nor a mystical philosophy, nor a vague sketch/proposal for a mathematical theory. Rather, it should be emphasized that

> *inter-universal Teichmüller theory is a rigorously formulated mathematical theory that has been verified countless times by quite a number of mathematicians, has undergone an exceptionally thorough seven and a half year long refereeing process, and was subsequently published in a leading international journal in the field of mathematics.*

[cf. the discussion of §1.1]. In particular, in order to avoid the sort of *vicious spirals* referred to above, it is of the utmost importance

> to concentrate, in discussions of inter-universal Teichmüller theory, on **substantive mathematical content**, as opposed to *non-mathematical* — such as *social*, *political*, or *psychological* — aspects or interpretations of the situation.

As discussed in §1.2, this is the main reason for the use of the term "RCS" in the present paper.

§**1.9.** **Geographical vs. mathematical proximity**

Historically, mathematical interaction between professional mathematicians relied on physical meetings or the exchange of hardcopy documents. Increasingly, however, advances in information technology have made it possible for mathematical interaction between professional mathematicians to be conducted electronically, by means of e-mail or online video communication. Of course, this does not imply that physical meetings or the exchange of hardcopy documents — especially in situations where physical meetings or the exchange of hardcopy documents do indeed **function** in a **meaningful** way, from the point of view of those involved — should necessarily be eschewed.

On the other hand, physical meetings between participants who live in distant regions requires **travel**. Moreover, travel, depending on the situations of the participants, can be a *highly taxing enterprise*. Indeed, travel, as well as lodging accommodations, typically requires the expenditure of a quite substantial amount of money, as well as **physical** and **mental effort** on the part of those involved. This *effort* can easily climb to *unmanageable* [i.e., from the point of view of certain of the participants] *proportions*, especially when *substantial cultural* — i.e., either in *mathematical* or in *non-mathematical* culture, or in *both* — *differences* are involved. The current situation involving the COVID-19 pandemic adds yet another dimension to the *reckoning*, from the point of view of the participants, of the physical and mental effort that must be expended in order to travel. As a result,

> *when, from the point of view of at least one of the key participants, the amount of* **effort**, **time**, *and/or* **money** *that must be expended to travel clearly exceeds, by a substantial margin — i.e., "$\ggg$" — the* **gain** *[i.e., relative to various mathematical or non-mathematical criteria of the key participant in question] that appears likely to be obtained from the travel under consideration, it is highly probable that the travel under consideration will end up simply not taking place.*

One "classical" example of this phenomenon "$\ggg$" is the relative scarcity of professional mathematicians in Europe or North America who travel to Japan frequently [e.g., at least once a year] or for substantial periods of time.

I have, at various times in my career, been somewhat surprised by assertions on the part of some mathematicians to the effect that travel should somehow be *forced* on mathematicians, i.e., to the effect that some sort of *coercion* may somehow *"override"* the *fundamental inequality* "$\ggg$" that exists as a result of the circumstances in which a mathematician finds him/herself in. In my experience, although this sort of coercion to travel may result in some sort of superficial influence in the very short term, *it can never succeed in the long term.* That is to say, the *fundamental circumstances* that give rise to the *fundamental inequality* "$\ggg$" can never be altered by means of such coercive measures to travel [cf. the discussion of [Rpt2014], (8)].

In this context, I was most impressed by the following two *concrete examples*, which came to my attention recently. In describing these examples, I have often used *somewhat indirect expressions*, in order to protect the privacy of the people involved.

Example 1.9.1: The insufficiency of geographical proximity. This example concerns the results obtained in a paper written in the fall of 2019 by a graduate student (St1) from country (Ct1). This student (St1) showed his paper to a prominent senior researcher (Pf1) at a university in country (Ct2) in a certain area of number theory. The education and career of this researcher (Pf1) was conducted entirely at universities in countries (Ct2), (Ct3), and (Ct4). This researcher (Pf1) informed (St1) of his very positive evaluation of the *originality* of the results obtained in the paper by (St1). Another prominent senior researcher (Pf2) in a certain area of number theory was informed by (Pf1) of the paper by (St1). This researcher (Pf2), who works at a university in country (Ct2) in close physical proximity to (Pf1), also took a generally positive position with regard to the paper by (St1). On the other hand, several months subsequent to this interaction between (Pf1) and (St1), a junior researcher (Pf3), who is originally from country (Ct5), but currently works at a university in country (Ct2) in close physical proximity to (Pf1) and (Pf2), informed student (St1) [via e-mail contacts between (Pf3) and (St1)'s advisor] that

> the results of the paper by (St1) are in fact **"well-known"** and **essentially contained in papers published in the 1990's by (Pf4)**, a prominent senior researcher in country (Ct6).

[To be more precise, in fact the results of the paper by (St1) are not entirely contained in the papers by (Pf4) in the sense that the paper by (St1) contains certain numerically explicit estimates that are not contained in the papers by (Pf4).] Country (Ct6) is in close physical proximity to country (Ct3), and in fact, one of the research advisors of researcher (Pf4), when (Pf4) was a graduate student, was a prominent researcher (Pf5) who is originally from country (Ct7), but has pursued his career as a mathematician mainly in countries (Ct3) and (Ct6). Here, it should be pointed out that (Pf1), (Pf2), and (Pf4) are very close in age, and that (Pf1) received his undergraduate education in country (Ct3) at one of the universities that played in prominent role in the career of (Pf5). The paper by student (St1) concerns mathematics that has been *studied extensively* by — and indeed forms one of the *central themes* of the research of — both (Pf1) and (Pf4), but from *very different points of view*, using *very different techniques*, since (Pf1) and (Pf4) work in substantially *different areas of number theory.* On the other hand, at *no time* during the initial several months of interaction between (Pf1), (Pf2), and (St1) was the work of (Pf4) mentioned. That is to say, (Pf1) and (Pf2) discussed the results obtained in the paper by (St1) in a way that can only be explained by the hypothesis that

> (Pf1) and (Pf2) were, at the time, **entirely unaware** of the very **close relationship** between the results obtained in the paper by (St1) and the

> **papers in the 1990's by (Pf4)**.

— i.e., despite the **numerous opportunities** afforded by **close physical proximity**, as well as proximity of age, for substantial interaction between (Pf1) and (Pf4). The paper by student (St1) is currently submitted for publication to a certain mathematical journal. Student (St1) recently received a referee's report for his paper, which apparently [i.e., judging from the comments made in the referee's report] was written by a mathematician working in an area of number theory close to (Pf1). This referee's report also makes *no mention* of the papers in the 1990's by (Pf4) and the fact that the results obtained in the paper by (St1) appear, with the exception of certain numerically explicit estimates, to be essentially contained in these papers of (Pf4). Finally, it should be mentioned that each official language of each of these countries (Ct1), (Ct2), (Ct3), (Ct4), (Ct5), (Ct6), (Ct7) belongs to the European branch of the Indo-European family of languages, and that at least six of the ten pairs of countries in the list (Ct2), (Ct3), (Ct4), (Ct6), (Ct7) share a common official language [i.e., with the other country in the pair].

<u>Example 1.9.2</u>: **The remarkable potency of mathematical proximity.** This example concerns the study of inter-universal Teichmüller theory by a graduate student (St2), who is originally from country (Ct8), but was enrolled in the doctoral program in mathematics at a university in country (Ct9) under the supervision of a senior faculty member (Pf6), who is originally from country (Ct10). This graduate student (St2) began his study of inter-universal Teichmüller theory as a graduate student and continued his study during his years as a graduate student with **essentially no mathematical contact** with any researchers who are significantly involved with inter-universal Teichmüller theory, except for his advisor (Pf6) and one mid-career researcher (Pf7) from country (Ct11). Here, we remark that the official language of each of these countries (Ct8), (Ct9), (Ct10), (Ct11) belongs to the European branch of the Indo-European family of languages. In particular, with the exception of a few very brief e-mail exchanges with me and a brief two-week long stay at RIMS in 2016 to participate in a workshop on IUT, this student (St2) had **essentially no mathematical contact**, prior to the fall of 2019, with **any researchers at Kyoto University** who are involved with inter-universal Teichmüller theory. Even in these circumstances,

> this student was able *not only* to achieve a **very technically sound understanding** of inter-universal Teichmüller theory on his own, by reading [IUTchI-IV] and making use of various resources, activities, and contacts within country (Ct9), *but also* to succeed, as a graduate student, in making **highly nontrivial original research contributions** to a certain mild generalization of inter-universal Teichmüller theory, as well as to certain related aspects of anabelian geometry.

My first [i.e., with the exception of a few very brief e-mail exchanges prior to this] mathematical contact with this student (St2) was in the fall of 2019. Although this student (St2) initially had some technical questions concerning aspects of inter-universal Teichmüller theory that he was unable to understand on his own, after a few relatively brief discussions in person with me, he was able to find answers to these technical questions in a relatively short period of time [roughly a month or two] without much trouble.

§1.10. Mathematical intellectual property rights

The **socio-political dynamics** generated by the proliferation of **logically unrelated fabricated versions** of inter-universal Teichmüller theory — of which RCS-IUT is perhaps the most frequently cited [cf. the discussion of §1.2, as well as Examples 2.4.5, 2.4.7, 2.4.8 below] — and further fueled by

- *grotesquely distorted mass media coverage and internet comments* [cf. the discussion of §1.8], as well as by
- the *conspicuous absence* of *detailed, explicit, mathematically substantive*, and *readily accessible written expositions* of the *logical structure* underlying the various central assertions of the proponents of such socio-political dynamics [cf. the discussion of §1.5],

have had the effect of **deeply disrupting** the normal process of absorption of inter-universal Teichmüller theory by the worldwide mathematical community. Left unchecked, this state of affairs threatens to pave the way for a field — i.e., the field of **mathematics** — **governed by socio-political dynamics** [cf. the discussion of (CmSn) in §1.5], rather than by *mathematical content.*

From a historical point of view, various forms of **institutional** and **conceptual infrastructure** — such as the notions of

- a **modern judiciary system**;
- **universal, inalienable human rights**;
- the **rule of law**;
- **due process of law**; and
- **burden of proof**

— were gradually developed, precisely with the goal of averting the outbreak of the sort of **socio-political dynamics** that were viewed as detrimental to society. In this context, it is interesting to note the *central role* played, for instance, in *courts of law*, by the practice of producing

> *detailed, explicit, logically substantive*, and *readily accessible* **written documentation** of the **logical structure** underlying the various central assertions of the parties involved.

This situation is very much reminiscent of the situation in *mathematics* discussed in §1.5 [cf. also the discussion of RCS-IUT in §1.3!], i.e., where we observe that it is **not even possible to analyze or debate**, in any sort of *meaningfully definitive* way, mathematical assertions — such as, for instance, the historically famous assertion of Fermat to the effect that he had a proof of "Fermat's Last Theorem", but did not write it down — in the **absence** of such **written documentation** of the **logical structure** of the issues under consideration.

From the point of view of the above discussion, it seems natural, in the case of mathematics, to introduce, especially in the context of issues such as the one discussed above involving *logically unrelated fabricated versions* of inter-universal Teichmüller theory, the notion of **mathematical intellectual property rights** [i.e., **"MIPRs"**]. As the name suggests, this notion is, in some sense, modeled on the conventional notion of *intellectual property rights* associated, for instance, with *trademarks* or *brand names* of corporations. In the case of this conventional notion, intellectual property rights may be understood as a tool for protecting the **"reliability"** or **"creditworthiness"** of trademarks or brand names of a corporation from the sort of **severe injury** to such trademarks or brand names that may ensue from the proliferation of **shoddy third-party imitations** of products produced by the corporation. Here, we observe that this "severe injury" often revolves around the creation of *severe obstacles* to the execution of activities that play a central role in the **operational normalcy** of the corporation.

Unlike this conventional notion, MIPRs should be understood as being associated — not to *corporations* or *individuals* for some *finite period of time*, but rather — to **mathematical notions** and **theories** and, moreover, are of *unlimited duration*. The purpose of MIPRs may be understood as the protection of the **"creditworthiness"** of such a mathematical notion or theory from the **severe injury** to the **operational normalcy** of mathematical progress related to notion/theory that ensues from the proliferation of **logically unrelated fabricated "fake" versions** of the notion/theory.

Before proceeding, we pause to consider one relatively elementary example of this notion of MIPRs.

<u>Example 1.10.1</u>: **The Pythagorean Theorem.**

(i) Recall the *Pythagorean Theorem* concerning the length of the hypotenuse of a right triangle in the Euclidean plane. Thus, if $0 < x \leq y < z \in \mathbb{R}$ are the lengths of the sides of a right triangle in the Euclidean plane, then the Pythagorean Theorem states that

$$x^2 + y^2 = z^2.$$

Various versions of this result apparently may be found not only in the writings of ancient Greece and Rome, but also in Babylonian, ancient Indian, and ancient Chinese documents. Well-known *"elementary proofs"* of this result may be obtained, for instance, by computing, in various equivalent ways, the *area* of suitable planar regions covered by right triangles or squares that are closely related to the given right triangle. On the other hand, such "elementary proofs" typically *do not address* the fundamental issue of how to *define* such notions as *length*, *angle*, and *rotation*, i.e., which are necessary in order to understand the *precise content* of the statement of the Pythagorean Theorem. Here, we observe that if, for instance, one tries to define the notion of the *length* of a line segment in Euclidean space in the conventional way, then the Pythagorean Theorem reduces, in effect, to a *meaningless tautology*! Moreover, although the notions of *length* and *angle* may be defined once one has defined the notion of a *rotation*, it is by no means clear how to give a natural definition of the notion of a rotation. For instance, one may attempt to define the notion of a *rotation* of Euclidean space as an element of the group generated by well-known matrices involving sines and cosines, but it is by no means clear that such a definition is *"natural"* or the *"right definition"* in some meaningful sense. Thus, in summary,

> it is by no means clear that such "elementary proofs" may be regarded as *genuine rigorous proofs* in the sense of modern mathematics.

(ii) From a modern point of view, a *natural, precise definition* of the fundamental notion of a **rotation** of Euclidean space [from which, as observed in (i), natural definitions of the notions of the notions of *length* and *angle* may be easily derived] may be given by thinking in terms of **invariant tensor forms** associated to **compact subgroups** of [the topological groups determined by] various **general linear groups**. From this modern point of view, the *precise form* of the Pythagorean Theorem [i.e., "$x^2 + y^2 = z^2$"] — and, in particular, the *significance of the "2" in the exponent!* — may be traced back to the theory of **Brauer groups** and the closely related **local class field theory** of the archimedean field "$\mathbb{R}$" of real numbers, i.e., in short, to various fundamental properties of the arithmetic of the **topological field** "$\mathbb{R}$".

(iii) Considering the situation discussed in (i), (ii), it is by no means clear, in any sort of *a priori* or *naive* sense, just why the Pythagorean Theorem should take the **precise form** "$x^2+y^2 = z^2$". Indeed, since this precise form of the Pythagorean Theorem continues to appear **utterly mysterious** even to numerous modern-day high school students — i.e., who grow up immersed in an environment replete with countless cultural links to modern mathematics, science, and technology! — it seems reasonable to assume that it should have appeared *all the more mysterious* to the individuals who populated the various ancient civilizations mentioned in (i). In particular, it is by no means unnatural to consider the possibility that assertions similar to the following assertions [stated relative to the notation introduced in (i)] might have been made by some hypothetical individual at some time in human history:

(Pyth1) "I don't understand why the relation in the Pythagorean Theorem is of the form '$x^2 + y^2 = z^2$', rather than '$x^2 \cdot y^2 = z^2$'."

(Pyth2) "I would like to investigate, in the context of the Pythagorean Theorem, whether or not the relation '$x^2 \cdot y^2 = z^2$' holds."

(Pyth3) "I investigated, in the context of the Pythagorean Theorem, whether or not the relation '$x^2 \cdot y^2 = z^2$' holds and discovered that there exist examples that show that this relation does not in fact hold in general."

(Pyth4) "The Pythagorean Theorem is **false** for the following reason: The Pythagorean Theorem states that '$x^2 \cdot y^2 = z^2$', but there exist counterexamples that show that this relation does not hold in general."

(iv) From the point of view of the discussion given above of MIPRs, the assertions (Pyth1), (Pyth2), (Pyth3) do *not* constitute a *violation* of the MIPRs of the Pythagorean Theorem, but rather are precisely the sorts of assertions/comments that *occur naturally in normal, sound research and educational activities* in mathematics. By contrast,

(VioMIPR) (Pyth4) may be regarded as a *classical example* of a **violation** of the **MIPRs** of the **Pythagorean Theorem**.

It is not difficult to imagine

(DtrVio) the **deeply detrimental effects** on the development of mathematics throughout history that would have occurred if **violations of MIPRs** similar to (Pyth4) regarding the Pythagorean Theorem arose and were **left unchecked**.

Moreover, in this context, it is important to observe that

(IgNoJst) assertions of **ignorance** of the **technical details** that one must understand in order to distinguish the *modified version* of the Pythagorean Theorem given in (Pyth4) from the *original version* of the Pythagorean Theorem do **not** by any means constitute a **justification** for participating in the **proliferation/citation/dissemination** of (Pyth4).

That is to say,

(BurPrf) the **burden of proof** of establishing any sort of logical relationship between such a *modified version* and the *original version* lies *exclusively* in the hands of the **proponents** of the modified version

[cf. the discussion immediately following the present Example 1.10.1]. Indeed, this notion of **burden of proof** (BurPrf) constitutes a *fundamental pillar* underlying the notion of **MIPRs** and may be readily understood by considering the corresponding [perhaps more familiar] situation surrounding the conventional notion of *intellectual property rights* as it is typically applied to technological devices such as *computers*: that is to say, an assertion of *technical ignorance* concerning the details of the internal technical structure of a *computer product* manufactured by *company A* [or *"A-product"* for short] and a *computer product* manufactured by *company X* [or *"X-product"* for short] does **not** by any means [i.e., *legal*, *ethical*, or otherwise!] **justify** the sale, by a [say, technically ignorant] computer dealer, of an X-product *advertised* as an *authentic A-product*.

(v) From a historical point of view, it appears to the author, in light of the discussion of (i), (ii), (iii), (iv), to be in some sense a sort of **miracle** that the Pythagorean Theorem was **"discovered"** in and, moreover, **survived** throughout the duration of numerous ancient civilizations, i.e., despite the fact that *dictatorial, authoritarian political regimes* with little regard for such modern notions as a *judiciary system* [in the modern sense], *inalienable human rights*, the *rule of law*, *due process of law*, *burden of proof*, and so on were by no means a rarity in the ancient world. Of course, to a certain extent, this situation may be understood as a consequence of the fact that the Pythagorean Theorem is closely related to the task of *direct measurement* of lengths of various easily accessed [i.e., even in the ancient world!] physical objects. From this point of view of "direct measurement", the "Pythagorean Theorem", as understood in various ancient civilizations, should perhaps be regarded [cf. the discussion of (i), (ii)] not so much as a result in *mathematics* [in the modern sense of the term], but rather as a *principally empirically substantiated* result in *physics*. Nevertheless, even when viewed from this point of view, it still seems like something of a *miracle* that this result *survived* throughout the duration of numerous ancient civilizations, unaffected by numerous *meaningless misunderstandings* of the sort discussed in (iii), (iv), especially considering that similarly meaningless misunderstandings of the Pythagorean Theorem continue to plague modern-day high school students!

Prior to the introduction above of the notion of **MIPRs**, this notion does not appear to have played an important role, at least in any sort of **explicit** sense, in discussions or analyses of the development of mathematics. On the other hand, Example 1.10.1 [cf., especially, Example 1.10.1, (iv)!] shows how, even at a *purely implicit* level,

> this notion of MIPRs has in fact played a **fundamentally important role** in the **development of mathematics** throughout history.

Of course, in the case of **violations** of the MIPRs of a mathematical notion or theory, conventional courts or judiciary systems are simply not equipped to play a meaningful role in dealing with such violations, since this would require an *extensive technical knowledge and understanding*, on the part of the judges or lawyers involved, of the *mathematics* under consideration. Indeed, it is useful to recall in this context that

- traditionally, any detrimental effects arising from such *violations of the MIPRs of a mathematical notion or theory* were typically averted in the field of mathematics by means of the **refereeing systems** of various well established mathematical journals;
- from the point of view of such *traditional refereeing systems* of well established mathematical journals, the **burden of proof** regarding the correctness of any *novel assertions* concerning existing mathematical notions or theories — such as, for instance, any assertions concerning some sort of **logical relationship** between a *modified version of a theory* and the *original version of the theory* — lies [not with the author of the original version of the theory (!), but rather] with the **author** of the manuscript containing the **novel assertions**.

On the other hand, as discussed in §1.8 [cf. also the discussion of §1.3], in the case of the quite **egregious MIPRs violations** constituted by *logically unrelated fabricated versions* of inter-universal Teichmüller theory, numerous *mass media reports* and *internet comments* released by individuals who are clearly *not* operating on the basis of a *solid, technically accurate understanding* of the mathematics involved are regarded, in certain sectors of the mathematical community, as carrying **much more weight** than an exceptionally thorough refereeing process in a well established mathematical journal by experts on the mathematics under consideration [cf. the discussion of Example 1.5.2]. This state of affairs is *deeply regrettable* and should be regarded as a **cause for alarm**. Perhaps in the long term, **new forms** of **institutional** or **conceptual infrastructure** may be developed for averting the *deeply detrimental* effects of this sort of situation. At the time of writing, however, it appears that

> the *only meaningful technical tool* currently available to humanity for dealing with this sort of situation lies in the production of *detailed, explicit, mathematically substantive, and readily accessible* **written expositions** of the **logical structure** underlying the assertions of the various parties involved [cf. the discussion surrounding (OvDlk) in §1.5, as well as the discussion of §1.12 below],

i.e., even when such assertions are purported to be a **"matter of course"** or **"common sense"**, that is to say, a matter that is so *profoundly self-evident* that any *"decent, reasonable observer"* would undoubtedly find such written documentations of logical structure to be **entirely unnecessary** [cf. the discussion surrounding (CmSn) in §1.5]. From a historical point of view, such *written documentations of logical structure* can then serve as a

valuable transgenerational or transcultural common core

of scholarly activity — a point of view that is reminiscent of the **logical relator AND "$\wedge$"**, which forms a *central theme* of the present paper.

§**1.11.** Social mirroring of mathematical logical structure

Discussions, on the part of some observers, concerning the situation surrounding inter-universal Teichmüller theory are often dominated by various **mutually exclusive** and **socially divisive/antagonistic dichotomies** [cf. also the related discussion of §1.8], i.e., such as the following:

(ExcDch) Is it the case that adherents of the RCS should be regarded as *mathematically correct/reliable/reasonable mathematicians*, while mathematicians associated with inter-universal Teichmüller theory should *not*, **OR** is it the other way around?

Here, the *"OR"* is typically understood as an **"XOR"**, i.e., *exclusive-or*. That is to say, such questions/dichotomies are typically understood as issues for which it can *never* be the case that an **"AND"** relation between the two possible alternatives under consideration holds. In discussions of mutually exclusive dichotomies such as (ExcDch), mathematicians associated with inter-universal Teichmüller theory, as well as the actual mathematical content of inter-universal Teichmüller theory, are often treated as **completely** and **essentially disjoint entities**, within the *international mathematical community*, from the mathematicians and mathematical research associated with the RCS.

In this context, it is interesting to note that this sort of *mutually exclusive dichotomy* is very much reminiscent of the **essential logical structure** of **RCS-IUT**, which, as discussed in the latter portion of Example 2.4.5 below, may be understood as being essentially logically equivalent to **OR-IUT**, as well as to **XOR-IUT**, i.e., to *logically unrelated fabricated versions* of inter-universal Teichmüller theory in which the crucial **logical AND "$\wedge$"** relation satisfied by the Θ-link of inter-universal Teichmüller theory is replaced by a **logical OR "$\vee$"** relation or, alternatively, by a **logical XOR "$\dot{\vee}$"** relation.

In fact, however,

(MthCnn) although it is indeed the case that the *international mathematical community*, as well as the *mathematical content* of the research performed by the international mathematical community, does *not* consist [in the language of classical algebraic geometry] of a *single irreducible component*, it is nevertheless undeniably **connected**.

In the case of inter-universal Teichmüller theory, this **abundant inter-connectivity** may be *explicitly witnessed* in the following aspects of the theory:

(IntCnn1) The mathematical content of various aspects of inter-universal Teichmüller theory is closely related to various *classical theories* such as the following:

- the **invariance** of **heights** of *abelian varieties* under **isogeny** [cf. the discussion of [Alien], §2.3, §2.4, as well as the discussion of Example 3.2.1 below; the discussion of §3.5 below];

- the classical proof in characteristic zero of the *geometric version* of the *Szpiro inequality* via the *Kodaira-Spencer morphism*, phrased in terms of the theory of **crystals** [cf. the discussion of [Alien], §3.1, (v), as well as the discussion of §3.5, §3.10 below];
- **Bogomolov's proof** over the complex numbers of the *geometric version* of the *Szpiro inequality* [cf. the discussion of [Alien], §3.10, (vi)];
- **classical complex Teichmüller theory** [cf. the discussion of Example 3.3.1 in §3.3 below];
- the classical theory of the **Jacobi identity** for the **theta function** [cf. the discussion of Example 3.3.2 in §3.3 below];
- the classical theory of the computation of the **Gaussian integral** via **polar coordinates** [cf. [Alien], §3.8].

We refer to [Alien], §4, for a more detailed discussion of such relationships between inter-universal Teichmüller theory and various classical mathematical theories.

(IntCnn2) In the context of the assertions of the RCS, it is important to recall [cf. the discussion of (MthVl) in Example 2.4.5, (viii), below] that [perhaps somewhat surprisingly!]

> in fact there is in some sense **no disagreement** among any of the parties involved with regard to the **mathematical validity** of the central mathematical assertions of the RCS

— i.e., so long as one **deletes** the **arbitrary label "inter-universal Teichmüller theory"** imposed by adherents of the RCS on the *logically unrelated fabricated versions* [i.e., RCS-IUT/OR-IUT/XOR-IUT] of inter-universal Teichmüller theory that typically appear in discussions of adherents of the RCS [cf. the discussion of §1.2, §1.3, §1.8, §1.10].

The situation discussed in (InnCnn2) is of particular interest in the context of the present paper since the *essential logical structure* of this situation discussed in (InnCnn2) — i.e.,

(CmMth) of a **common mathematical understanding** of the *mathematical validity* of the various assertions under discussion, so long as one keeps track of the **distinct labels** *"inter-universal Teichmüller theory"* and *"RCS-IUT/OR-IUT/XOR-IUT"*

— is *remarkably similar* to the *essential logical structure* of the situation surrounding the *central theme* of the present paper, namely, the crucial **logical AND "$\wedge$"** property of the Θ-link in inter-universal Teichmüller theory — cf. the discussion of §2.4, §3.4 below. At a more elementary level, (CmMth) may be understood as being qualitatively essentially the *same phenomenon* as the phenomenon discussed in Example 1.10.1, (iii), i.e., the distinction between (Pyth3) [where the *distinct* *label "Pythagorean Theorem"* is treated properly] and (Pyth4) [where the *distinct label "Pythagorean Theorem"* is *not* treated properly].

§1.12. Computer verification, mathematical dialogue, and developmental reconstruction

One *question* that is frequently posed, in the context of the *entirely unnecessary confusion* that results from the plethora of **misinformation** and **false narratives** concerning inter-universal Teichmüller theory in the English-language mass media and internet [cf. the discussion of §1.8, §1.10], is the following:

(CmpVer) *Why not use computers to verify the mathematical validity of inter-universal Teichmüller theory?*

The implication here is that *"computers"* may be regarded as an entity inherently endowed with a sort of *impeccable neutrality/impartiality* with regard to the verification of mathematical assertions.

The fundamental problem with (CmpVer) lies in the *essentially tautological observation* that

(Algor) **no** computer verification algorithm for verifying some mathematical assertion can yield a verification of the **validity** of the **algorithm itself**, i.e., of the **presumed relationship** between

- the *mechanical output* yielded by the algorithm and
- the *conventional human sense of "mathematical correctness"*

— i.e., a relationship on whose integrity any sort of computer verification must be premised.

Of course, in situations where the issue raised in (Algor) is not an issue of concern — i.e., such as situations involving *routine numerical computations* or *manipulation of data* in some *relatively simple combinatorial framework* [such as a finite group, a finite simplicial complex, or a finite chain of Boolean operators] — computer verification can indeed function as a meaningful tool for the verification of mathematical assertions.

On the other hand, in situations — such as the situation surrounding such **logically unrelated fabricated versions** of inter-universal Teichmüller theory as RCS-IUT — in which the central issue lies [cf. the discussion of §1.2, §1.3, §1.10, as well as Example 2.4.5, 2.4.7, 2.4.8, below] in the *erroneous confusion* of such *logically unrelated fabricated versions* of a theory with the original version of the theory, computer verifications can **never** *yield meaningful or substantive progress*, since the erroneous confusion of logically unrelated fabricated versions of a theory with the original version of the theory *completely invalidates*, in a very essential and inevitable way, the *"presumed relationship"* discussed in (Algor) on which any sort of computer verification of a mathematical assertion must be premised. More elementary examples of this phenomenon of *erroneous confusion of logically unrelated fabricated versions of a theory with the original version of the theory* may be seen in

- the situation surrounding the assertion of Example 1.10.1, (iii), (Pyth4);
- the situation surrounding the [*erroneous*!] point of view described in

> Example 2.1.1 concerning the classical theory of integration on the real line

— i.e., situations that arise from *essentially non-mathematical [e.g., social/political/psychological] circumstances* and, as a result, are *clearly not amenable to resolution via computer verification.*

In the context of (CmpVer), it is perhaps of interest to recall from the Introduction [cf. also the discussion at the beginning of §3.10] the point of view — motivated by the well-known *functional completeness*, in the sense of *propositional calculus*, of the collection of *Boolean operators* consisting of *logical AND "$\wedge$"*, *logical OR "$\vee$"*, and *negation "$\neg$"* — that one can, in principle, express

> the **essential logical structure** of any **mathematical argument** or **theory** in terms of **elementary logical relations**, i.e., such as **logical AND "$\wedge$"**, **logical OR "$\vee$"**, and **negation** "$\neg$".

Indeed, it is precisely this point of view that formed the *central motivation* and *conceptual starting point* of the exposition given in the present paper concerning the **essential logical structure** of inter-universal Teichmüller theory, which may be represented **symbolically** as follows:

$$\begin{array}{rcl} A \wedge B & = & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ \ldots) \\ & \Longrightarrow & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ \ldots \ \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \ \ldots) \\ & \Longrightarrow & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ \ldots \ \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \ \ldots \ \dot{\vee} B''_1 \dot{\vee} B''_2 \dot{\vee} \ \ldots) \\ & & \vdots \end{array}$$

That is to say, in summary:

(SymIUT) The **symbolic representation** [cf. the above display!] of the *essential logical structure* of inter-universal Teichmüller theory exposed in the present paper may be understood as being, in some sense, the *closest realistic approach* to the **essential spirit** of (CmpVer). Moreover, this symbolic representation [cf. the above display!] is sufficiently *simple* and *transparent* that, *once it has been properly communicated*, it may be verified readily by mental computation in a matter of minutes *without the use of a computer*!

Here, we note that the *relative simplicity* of this *symbolic representation* of (SymIUT) is obtained as a result of organizing/compartmentalizing into **"blackboxes"** various *"blocks"* of inter-universal Teichmüller theory that consist of *anabelian geometry* or the theory of *étale theta functions*, and whose validity has never arisen as a matter of discussion.

Ultimately, misunderstandings resulting from logically unrelated fabricated versions can only be overcome by studying the *original papers* [IUTchI-IV] [cf. also [Alien], as well as the present paper!] on inter-universal Teichmüller theory, or, if this is not sufficient, by engaging in **constructive mathematical dialogue** with mathematicians who do have a substantial, accurate understanding of the theory [cf. the discussion of §1.4, §1.5, §1.6]. Indeed, perhaps more to the point, there appears to be a *conspicuous tendency*, in certain sectors of the mathematical community, to

(RfsDlg) utilize the **proliferation/citation** of **logically unrelated fabricated versions** of the theory as a sort of **"lame excuse"/subterfuge** to justify a stance of **refusal** *to engage in such constructive mathematical dialogue* concerning the theory

— cf. the discussion of §1.3, §1.10, especially the discussion of (VioMIPR), (DtrVio), (IgNoJst), (BurPrf) in Example 1.10.1, (iv); the discussion of Examples 3.10.1, 3.10.2 below. Moreover,

(DngPrc) the situation described above in (RfsDlg), if left *unchecked*, constitutes, in the long-term, a **dangerous precedent** from the point of view of maintaining a state of **operational normalcy** in the field of mathematics in a fashion consistent with such fundamental democratic principles as the **rule of law**, **due process of law**, and **burden of proof**

— cf. the discussion of §1.10, especially the discussion of (VioMIPR), (DtrVio), (IgNoJst), (BurPrf) in Example 1.10.1, (iv).

Nevertheless, in this context, it is also of fundamental importance to recall

(OvrMs) the existence of numerous mathematicians [of many diverse nationalities!] who were indeed successful in **overcoming** various **meaningless misunderstandings** concerning inter-universal Teichmüller theory precisely by persistently engaging in **constructive mathematical dialogue** concerning the theory.

Such dialogues typically involve the painstaking and time-consuming process of

(ExplLS) sifting through and analyzing the assertions of the mathematician in question concerning inter-universal Teichmüller theory in order to make **precise** and **explicit** the exact content of the **logical structure** *underlying such assertions* [cf. the discussion of §1.5, especially Examples 1.5.1, 1.5.2].

On the other hand, it is important to emphasize that such efforts, when seen through to their conclusion, have always led to a situation in which the mathematician in question realizes his/her misunderstandings of the theory and ultimately concedes that, at least so far as he/she can see,

(MthVlTh) there is indeed **no mathematical reason to deny the mathematical validity** of the theory.

That is to say, this chain of events is precisely the chain of events that should be expected from any constructive mathematical dialogue carried out in a suitable, sincere, and rational fashion concerning a rigorously formulated mathematical theory.

Thus, in summary, at least in a direct, literal sense, *ultimately, the only way to overcome meaningless misunderstandings* of the theory that arise from *logically unrelated fabricated versions of the theory* is to

(CfrMth) **directly confront** the **mathematical content** of the original theory, either

- by studying the *original papers* [IUTchI-IV] [cf. also [Alien], as well as the present paper!] on inter-universal Teichmüller theory, or, if this is not sufficient,
- by engaging in *constructive mathematical dialogue* with mathematicians who do have a substantial, accurate understanding of the theory.

[cf. the discussion of §1.3, §1.10, as well as Examples 3.10.1, 3.10.2 below].

Conversely,

(BlkAcc) a stance of *systematic* and *institutionally endorsed refusal to confront the mathematical content of the original theory* has the effect of orchestrating the creation of a sort of artificial **blackhole** relative to the issue of **mathematical accountability** [cf. the discussion of [EMSCOP] in §1.3] and leads to the sort of **absurd pathologies** and **obstructions** to the **operational normalcy** of the field of mathematics discussed in detail in §1.10 [cf., especially, Example 1.10.1, (iii), (Pyth4)].

That is to say, unlike the **IUT community**, which bears **active mathematical responsibility** for the mathematical content of inter-universal Teichmüller theory in the **long-term** by maintaining an extensive, sustained **infrastructural apparatus** of mathematical activities surrounding inter-universal Teichmüller theory — such as

- *workshops*, of one to two weeks in length, concerning inter-universal Teichmüller theory [e.g., at RIMS in March 2015, in Oxford in December 2015, at RIMS in July 2016, and at RIMS in September 2021] and *numerous lecture series* [e.g., in Kumamoto in May 2014, at RIMS in December 2015, in Yokohama in November 2018, and at RIMS in December 2021], which have led to a quite substantial stock of *widely and readily accessible PDF files of slides* and *videos of lectures* exposing inter-universal Teichmüller theory;
- *extensive one-to-one mathematical discussions* between mathematicians all over the world concerning inter-universal Teichmüller theory via *e-mail* and *online video meetings*;
- *joint research projects* concerning the further development of inter-universal Teichmüller theory [cf., e.g., [ExpEst]]

— the author remains *entirely unable*, despite *years of intensive effort* [cf. the discussion of §1.3, §1.5, §1.6], to locate *even a single mathematician* who is willing to bear **active responsibility** for the **mathematical content** of **RCS-IUT** by engaging in similar mathematical activities/mathematical dialogue. The *fundamental* **qualitative** *difference* constituted by this **egregrious absence** of an infrastructural apparatus supporting the assertions of the RCS — that is to say, put another way, this sort of **"hit-and-run"/"dead-end"** approach to making vaguely formulated mathematical assertions that are not supported by detailed documentation/exposition apparatuses, i.e., in *violation* of the [EMSCOP] [cf. the discussion of the [EMSCOP] in §1.3] — is precisely what is meant by the notion of a **"blackhole"** of **mathematical accountability** discussed in (BlkAcc).

On the other hand, from a more *long-term, historical point of view*, it is perhaps of interest to observe that there is another approach to witnessing the validity of a mathematical theory, namely,

(DvpRcn) the approach of **developmental reconstruction**, i.e., of *"reconstructing the validity"* of a mathematical theory by witnessing the **subsequent developments** that ensue from the theory.

This point of view is particularly of interest in the context of inter-universal Teichmüller theory, given the *central role* played in inter-universal Teichmüller theory by **anabelian geometry**, i.e., which revolves around the development of **reconstruction algorithms** that allow one to reconstruct *conventional algebraic structures* [i.e., of the sort that typically appear in algebraic/arithmetic geometry] from *more primitive combinatorial structures* such as topological groups.

This approach of *developmental reconstruction* may be applied, for instance, to the task of evaluating the level of mathematical or scientific development of **ancient civilizations**, i.e., *not* via the direct study of detailed theoretical expositions [which are typically *not readily available* — cf. the discussion of §1.5!] of the mathematics or science understood by such an ancient civilization, but rather by observing what may be understood as the *"fruits"* of this mathematics or science, e.g., in the form of *architectural achievements* such as the famous

- **pyramids of Egypt** or
- **Nazca lines** and **mysterious ruins of Puma Punku** in South America.

Another important [though non-architectural!] example of this sort of phenomenon may be seen in

- the list of **Pythagorean triples** in the famous **Babylonian tablet Plimpton 322**

— i.e., which is *particularly notable* in that it strongly suggests [that is to say, despite the fact that it is not accompanied by any sort of *explicit theoretical exposition*!] an understanding of *algebraic manipulation* on a par with the essential content of Example 1.12.1 below.

Example 1.12.1: Explicit parametrization of Pythagorean triples. The set of *integral solutions* — i.e., solutions in the ring of integers $\mathbb{Z}$, also known as *Pythagorean triples* — of the equation $x^2+y^2 = z^2$ may be parametrized by applying the substitutions $\frac{x}{z} \mapsto u$, $\frac{y}{z} \mapsto v$ and considering the set of *rational solutions* — i.e., solutions in the field of rational numbers $\mathbb{Q}$ — of the equation $u^2 + v^2 = 1$. The solutions of this last equation $u^2 + v^2 = 1$ in $\mathbb{Q}$ — or, indeed, in any field of characteristic $\neq 2$ — may be *completely parametrized* by the substitutions

$$u \mapsto \tfrac{t^2-1}{t^2+1}, \quad v \mapsto \tfrac{2t}{t^2+1}$$

— where we observe that

$$\left(\tfrac{t^2-1}{t^2+1}\right)^2 + \left(\tfrac{2t}{t^2+1}\right)^2 \ = \ 1, \qquad t \ = \ \tfrac{v}{1-u} \ = \ \tfrac{u+1}{v} \ = \ \left\{\left(\tfrac{t^2-1}{t^2+1}\right) \ + \ 1\right\} \cdot \left(\tfrac{2t}{t^2+1}\right)^{-1},$$

$$2 \cdot \left(t + \tfrac{1}{t}\right)^{-1} \;=\; 2 \cdot \left(\tfrac{u+1}{v} + \tfrac{v}{u+1}\right)^{-1} \;=\; 2 \cdot \left(\tfrac{(u+1)^2+v^2}{v(u+1)}\right)^{-1}$$

$$=\; 2 \cdot \left(\tfrac{u^2+v^2+1+2u}{v(u+1)}\right)^{-1} \;=\; 2 \cdot \left(\tfrac{2(u+1)}{v(u+1)}\right)^{-1} \;=\; v$$

That is to say, this *parametrization by* t [for t such that $t^2+1 \neq 0$] gives a *complete list* of all solutions of the equation $u^2+v^2 = 1$ [for u such that $u-1 \neq 0$] in any field of characteristic $\neq 2$ [or, indeed, by interpreting "$\neq 0$" as a condition of invertibility, in any ring in which 2 is invertible].

On the other hand, in this context, it is of interest to note that, at least as of the time of writing of the present paper,

(AncBas) *no ancient civilization has produced evidence of knowledge of the equation*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; \frac{\pi^2}{6}$$

— i.e., of the solution of the so-called *Basel problem.*

That is to say, here we note that this observation (AncBas) is *valid* despite the fact that each of the *essential components* of this equation — i.e.,

· the *positive integers of "sufficiently large value"*,

· the elementary operations of *addition/multiplication/division*,

· the notion of the *length of the circumference of a circle of radius* 1, and

· the idea of a sum of [real] numbers coming *arbitrarily close, up to a very small margin of error,* to another [real] number

— may be readily *expressed* in terms understandable to many advanced ancient civilizations. Of course, the discovery of an ancient civilization that produced evidence of some sort of knowledge of the equation in the display of (AncBas) would be quite **startling** since it would strongly suggest [that is to say, even if it is not accompanied by any sort of *explicit theoretical exposition*!] an understanding of numerous *ideas and theorems* not only from elementary differential and integral calculus, but also possibly from **Fourier analysis on the circle** and **complex analysis on the complex plane**.

Finally, we return to our discussion of inter-universal Teichmüller theory. In the case of inter-universal Teichmüller theory, the phenomenon of **developmental reconstruction** may already be seen, albeit in a relatively weak sense, in the *numerical results* of [ExpEst]. Stronger examples of this phenomenon may be seen, however, in various **enhanced versions** of inter-universal Teichmüller theory that are currently under development, which are expected to give rise to various new types of applications of inter-universal Teichmüller theory. Such enhanced versions suggest strongly that the original version of inter-universal Teichmüller theory given in [IUTchI-IV] [cf. also [Alien], as well as the present paper!] should perhaps be regarded as being only the first example of a much larger collection of examples of **"anabelian adèlic analysis"**, i.e., in the spirit of the point of view that the various types of *prime-strips* that occur in inter-universal Teichmüller theory may be thought of [cf. the discussion at the end of [Alien], §3.3, (iv)] as a sort of anabelian/monoid-theoretic version of the classical notion of *adèles/idèles* that appears throughout conventional arithmetic geometry and number theory. Here, we observe that this term *"anabelian adèlic analysis"* is of interest from a *historical point of view* in that it encapsulates, in a perhaps surprisingly efficient fashion, a quite substantial portion of the *historical development of arithmetic geometry* that underlies significant portions of inter-universal Teichmüller theory: indeed,

(AAA0) the *non-holomorphic (!)* **"analysis"** on the *real line* surrounding *Euler's formula* [cf. the discussion of §1.5], *Euler's solution of the Basel problem* [cf. the discussion above of (AncBas)], the *gamma function*, and the *Gaussian integral* [also known as the *Euler-Poisson integral* — cf. also the discussion of [Alien], §3.8] — i.e., analysis of the sort practiced during the **18-th century** by such mathematicians as *Euler* and the *Bernoullis* — may be understood as a sort of *essential preparatory phase* that paved the way for the *holomorphic* analysis/function theory of the 19-th century discussed in (AAA1), below, surrounding the *theta* and *zeta functions* [as well as the *complex logarithm* — cf. the discussion of §1.5];

(AAA1) at a more genuine/non-preparatory level, **"analysis"** may be regarded as referring to the **classical complex function theory** — pioneered by such **19-th century** mathematicians as *Jacobi*, *Riemann*, and *Mellin* — on the complex plane/upper half-plane surrounding the well-known *functional equations* of the *theta function* and the *Riemann zeta function*, which are related via the *Mellin transform* [cf. the discussion of §1.5];

(AAA2) **"adèlic"** may be understood as referring to the **adèlization** — developed by such **20-th century** mathematicians as *Chevalley*, *Weil*, *Iwasawa*, *Artin*, *Tate*, and *Langlands* — of the classical complex function theory of (AAA1), a development which led, in particular, to the well-known adèlic proof of the *functional equation* of the *Dedekind zeta function* [originally due to *Hecke*] and subsequently to the *representation-theoretic approach* of the *Langlands program*;

(AAA3) **"anabelian"** may be interpreted as referring to the **"anabelianization"** of the *classical functional equation of the theta function on the upper half-plane* in the fashion of *inter-universal Teichmüller theory*, i.e., by relating *Galois groups/arithmetic fundamental groups* to *ring/field theory* — **not** via representation theory, as in (AAA2) (!), but rather — by applying *cyclotomic rigidity isomorphisms* and *Kummer theory* to relate the *étale-like* objects constructed by means of *anabelian* algorithms to their *Frobenius-like* counterparts arising from ring/field theory.

Thus, (AAA2) may be understood as an approach to the **"arithmetization"** of the classical function theory of (AAA1) by means of **adèlization/representation theory**, i.e., by thinking, in short, of the *adèles* as a new domain [i.e., more precisely, locally compact topological space of uncountable cardinality] in which to conduct *analysis/function theory.* By contrast, (AAA3) may be interpreted as an approach to the *"arithmetization"* of the classical function theory of (AAA1) by thinking of the **abstract topological groups** that arise as **absolute Galois**

groups/arithmetic fundamental groups as the natural domain [i.e., more precisely, locally compact topological space of uncountable cardinality] for conducting *analysis/function theory.* Thus, in summary, at the level of *natural domains* for *conducting the analysis/function theory surrounding the theta/zeta functions*, one can discern a **fascinating historical progression**

$$(\mathrm{AAA0}) \quad \rightsquigarrow \quad (\mathrm{AAA1}) \quad \rightsquigarrow \quad (\mathrm{AAA2}) \quad \rightsquigarrow \quad (\mathrm{AAA3})$$

corresponding to

$$\mathbb{R} \quad \rightsquigarrow \quad \mathbb{C} \quad \rightsquigarrow \quad \textbf{adèles} \quad \rightsquigarrow \quad \textbf{arithmetic fundamental groups}.$$

In closing, we note that, in some sense, this interpretation of (AAA3) is consistent with the *spirit* of *Grothendieck's anabelian philosophy* as an approach to *diophantine geometry* via anabelian geometry, although, as discussed in [IUTchI], §I5, whereas Grothendieck apparently envisaged this approach as centering around the *Section Conjecture*, the anabelian geometry that actually appears in inter-universal Teichmüller theory consists mainly of **absolute anabelian geometry** over *number fields* and *p-adic local fields.*

Section 2: Elementary mathematical aspects of "redundant copies"

The *essence* of the central mathematical assertions of the RCS revolves, perhaps somewhat remarkably, around *quite elementary considerations* that lie well within the framework of undergraduate-level mathematics. Before examining, in §3, the assertions of the RCS in the technical terminology of inter-universal Teichmüller theory, we pause to give a detailed exposition of these elementary considerations.

§**2.1.** <u>The history of limits and integration</u>

The classical notion of **integration** [e.g., for continuous real-valued functions on the real line], as well as the more fundamental, but closely related notion of a **limit**, have a long history, dating back [at least] to the 17-th century. Initially, these notions did not have rigorous definitions, i.e., were not *"well-defined"*, in the sense understood by mathematicians today. The lack of such rigorous definitions frequently led, up until around the end of the 19-th century, to **"contradictions"** or **"paradoxes"** in mathematical work — such as *Grandi's series*

$$\sum_{n=0}^{\infty} (-1)^n$$

— concerning integrals or limits.

Ultimately, of course, the theory of limits and integrals *evolved*, especially during the period starting from around the mid-19-th century and lasting until around the early 20-th century, to the extent that such *"contradictions/paradoxes"* could be resolved in a definitive way. This process of *evolution* involved, for instance, in the case of integration, first the introduction of the *Riemann integral* and later the introduction of the *Lebesgue integral*, which made it possible to integrate functions

— such as, for instance, the *indicator function* on the real line of the *subset of rational numbers* — whose Riemann integral is not well-defined.

Here, it should be noted that at various key points during this *evolution* of the notions of limits and integration, the central *"contradictions/paradoxes"* that, at times, led to substantial criticism and confusion arose from a **solid, technically accurate understanding** of the content and logical structure of the assertions — such as, for instance, various possible approaches to computing the value of Grandi's series — at the center of these "contradictions/paradoxes". It is precisely for this reason that such criticism and confusion ultimately lead to substantive refinements in the theory that were sufficient to resolve the original "contradictions/paradoxes" in a definitive way.

> Such *constructive episodes in the history of mathematics* — which may be studied by scholars today precisely because of the existence of *detailed, explicit, mathematically substantive, and readily accessible written records*! [cf. the discussion of §1.5] — stand in *stark contrast* to [cf. the discussion of (UndIg) in §1.3] criticism of a mathematical theory that is based on a **fundamental ignorance** of the content and logical structure of the theory, such as the following *"false contradiction"* in the theory of integration, which may be observed in some students who are still in an initial stage with regard to their study of the notion of integration.

Example 2.1.1: False contradiction in the theory of integration. Consider the following computation of the definite integral of a real-valued function on the real line

$$\int_0^1 x^n \, dx \; = \; \tfrac{1}{n+1}$$

for n a positive integer. Suppose that one takes the [drastically oversimplified and manifestly absurd, from the point of view of any observer who has an accurate understanding of the theory of integration!] point of view that the *most general possible interpretation* of the equation of the above display is one in which the following three *symbols*

$$\text{“}\int_0^1\text{”}, \quad \text{“}x\text{”}, \quad \text{“}dx\text{”}$$

are allowed to be *arbitrary positive real numbers* a, b, c. Here, we note that in the case of "dx", such a substitution "$dx \mapsto c$" could be *"justified"* by quoting *conventional "ϵ-δ" treatments* of the theory of limits and integrals, in which *infinitesimals* — i.e., such as "dx" or "ϵ" — are allowed to be *arbitrary positive real numbers*, which are regarded as being *"arbitrarily small"*, and observing that any positive real number c is indeed much smaller than "most other positive real numbers" [such as $1000 \cdot c$, etc.]. On the other hand, by substituting the values $n = 1, 2, 3$, one obtains relations

$$abc = 1, \quad ab^2c = \tfrac{1}{2}, \quad ab^3c = \tfrac{1}{3}.$$

The first two of these relations imply that $b = \frac{1}{2}$ [so $b^2 = \frac{1}{4}$], while the first and third relations imply that $b^2 = \frac{1}{3} \neq \frac{1}{4}$ — a *"contradiction"*!

§**2.2.** **Derivatives and integrals**

In the context of the historical discussion of integration in §2.1, it is interesting to recall the **fundamental theorem of calculus**, i.e., the result to the effect that, roughly speaking, the operations of *integration* and *differentiation* of functions [i.e., real-valued functions on the real line satisfying suitable conditions] are *inverse* to one another. Thus, from a certain point of view,

> the *"essential information"* contained in a *function* may be understood as being **"essentially equivalent"** to the *"essential information"* contained in the *derivative* of the function

— that is to say, since one may always go back and forth at will between a function and its derivative by integrating and then differentiating. This point of view might then tempt some observers to conclude that

> any mathematical proof that relies, in an essential way, on consideration of the *derivative* of a function must be *fundamentally flawed* since any information that might possibly be extracted from the derivative of the function should already be available [cf. the *"essential equivalence"* discussed above] from the function prior to passing to the derivative, i.e., in *"contradiction"* to the essential dependence of the proof on passing to the derivative.

Alternatively, this point of view may be summarized in the following way:

> the *"essential equivalence"* discussed above implies that any usage, in a mathematical proof, of the derivative of a function is necessarily inherently **redundant** in nature.

In fact, of course, such "pseudo-mathematical reasoning" is itself **fundamentally flawed**. Two examples of well-known proofs in arithmetic geometry that depend, in a essential way, on passing to the derivative will be discussed in the final portion of §3.2 below [cf. (InvHt), (FrDff)]. These examples are in fact *closely related to the mathematics that inspired* inter-universal Teichmüller theory [cf. the discussion in the final portion of §3.2 below]. One central aspect of the situations discussed in §3.2 below is the exploitation of properties of [various more abstract analogues of] the *derivative of a function* [cf., e.g., the discussion of Example 3.2.1, (vii), below] that **differ**, in a very **substantive, qualitative** way, from the properties of the original function. One important example of this sort of situation is the *validity/invalidity* of various **symmetry properties**. This phenomenon may be observed in the following elementary example.

Example 2.2.1: Symmetry properties of derivatives.

(i) The real-valued function

$$f(x) = x$$

on the real line is *not invariant* [i.e., *not symmetric*] with respect to translations by an arbitrary constant $c \in \mathbb{R}$. That is to say, in general, it is not the case that $f(x + c) = f(x)$. On the other hand, the derivative

$$f'(x) = 1$$

of this function is **manifestly invariant/symmetric** with respect to such translations.

(ii) One variant of the discussion of (i) is the following example, which, in some sense, illustrates the *essential spirit* of *differential and integral calculus.* Consider, for some positive integer n and positive real number λ, a real-valued function $f : \{0, 1, \ldots, n\} \rightarrow \mathbb{R}$ on the set of nonnegative integers $\leq n$ such that, for each $i \in \{0, 1, \ldots, n-1\}$,

$$|f(i+1) - f(i)| \ \leq \ \lambda.$$

Then one may *approximate* $f(n)$ in terms of $f(0)$ — **not** (!) by **arbitrarily identifying** the elements $0, n \in \{0, 1, \ldots, n\}$ or the elements $f(0), f(n) \in \mathbb{R}$, but rather — by adding up the **possible variations** of the function f as one gradually increases i from 0 to n as follows:

$$|f(n) - f(0)| \ \leq \ n \cdot \lambda.$$

This very elementary example may be understood as a *faithful representation* of the [again *very elementary*!] *set-theoretic foundational apparatus* underlying inter-universal Teichmüller theory [cf. the discussion surrounding (EngIf) in §3.10 below]. That is to say, the *"possible variations"* in the above discussion correspond to the *"fuzzifications"* — i.e., *indeterminacies* — that appear in the discussion surrounding (EngIf) in §3.10 below, while the *values* "$f(0)$", "$f(n)$" [or, alternatively, "$f(n)$", "$f(0)$", if one prefers!] correspond, respectively, to the *log-volumes* of the *q-pilot* and Θ-*pilot objects* in the codomain and domain of the Θ-*link* of inter-universal Teichmüller theory.

§**2.3.** <u>**Line segments vs. loops**</u>

By comparison to the examples given in §2.1, §2.2, the following *elementary geometric examples* are much more *closely technically related* to the assertions of the RCS concerning inter-universal Teichmüller theory.

<u>Example 2.3.1</u>: Endpoints of an oriented line segment.

(i) Write

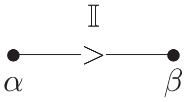
$$\mathbb{I} \ \stackrel{\mathrm{def}}{=} \ [0, 1] \ \subseteq \ \mathbb{R}$$

for the *closed unit interval* [i.e., the set of nonnegative real numbers ≤ 1] in the real line $\mathbb{R}$. Thus, $\mathbb{I}$ is equipped with a natural *topology* [i.e., induced by the topology of $\mathbb{R}$], hence can be regarded as a *topological space*, indeed more specifically, as a *one-dimensional topological manifold with boundary* that is equipped with a natural *orientation* [i.e., induced by the usual orientation of $\mathbb{R}$]. Write

$$\alpha \ \stackrel{\mathrm{def}}{=} \ \{0\}, \quad \beta \ \stackrel{\mathrm{def}}{=} \ \{1\}$$

for the topological spaces [consisting of a single point!] determined by the two endpoints of $\mathbb{I}$. Thus, α and β are *isomorphic as topological spaces.* In certain situations that occur in category theory, it is often customary to replace a given category by a full subcategory called a *skeleton*, which is *equivalent* to the given category, but also satisfies the property that any two isomorphic objects in the skeleton are equal. This point of view of working with *skeletal categories* [i.e., categories which are their own skeletons] is motivated by the idea that *nonequal isomorphic objects* are *"redundant"*. Of course, there are indeed various situations in which nonequal isomorphic objects are redundant in the sense that working with skeletal categories, as opposed to arbitrary categories, does not result in any substantive difference in the mathematics under consideration.

(ii) On the other hand, if, in the present discussion of $\mathbb{I}$, α, β — which one may visualize as follows



— one **identifies** α and β, then one obtains a *new topological space*, that is to say, more specifically, an *oriented one-dimensional topological manifold* [whose orientation is induced by the orientation of $\mathbb{I}$]

$$\mathbb{L} \overset{\text{def}}{=} \mathbb{I}/\langle \alpha \sim \beta \rangle$$

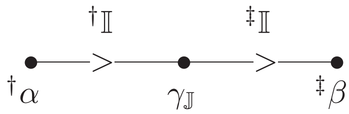
that is homeomorphic to the unit circle, i.e., may be visualized as a **loop**. Write $\gamma_{\mathbb{L}} \subseteq \mathbb{L}$ for the image of $\alpha \subseteq \mathbb{I}$, or, equivalently, $\beta \subseteq \mathbb{I}$, in $\mathbb{L}$. As is well-known from elementary topology, the topological space $\mathbb{L}$ is *structurally/qualitatively very different* from the topological space $\mathbb{I}$. For instance, whereas $\mathbb{I}$ has a *trivial fundamental group*, $\mathbb{L}$ has a *nontrivial fundamental group* [isomorphic to the additive group of integers $\mathbb{Z}$]. In particular,

> it is *by no means* the case that the fact that α and β are isomorphic as topological spaces implies a sort of **"redundancy"** to the effect that any mathematical argument involving $\mathbb{I}$ [cf. the above observation concerning fundamental groups!] is **entirely equivalent** to a corresponding mathematical argument in which α and β are **identified**, i.e., in which "$\mathbb{I}$" is *replaced* by "$\mathbb{L}$".

(iii) In this context, we observe that the [one-dimensional oriented topological manifold with boundary] $\mathbb{I}$ does **not** admit any **symmetries** that *switch* α and β. Moreover, even if one passes to the quotient $\mathbb{I} \twoheadrightarrow \mathbb{L}$, the [one-dimensional oriented topological manifold] $\mathbb{L}$ does **not** admit any **symmetries** that *reverse the orientation* of $\mathbb{L}$.

Example 2.3.2: Gluing of adjacent oriented line segments.

(i) A similar elementary geometric situation to the situation discussed in Example 2.3.1, but which is technically a bit more similar to the situation that arises in inter-universal Teichmüller theory may be given as follows. We begin with **two distinct copies** ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$ of $\mathbb{I}$. Thus, ${}^{\dagger}\mathbb{I}$ has *endpoints* ${}^{\dagger}\alpha$, ${}^{\dagger}\beta$ [i.e., corresponding respectively to the endpoints α, β of $\mathbb{I}$]; similarly, ${}^{\ddagger}\mathbb{I}$ has *endpoints* ${}^{\ddagger}\alpha$, ${}^{\ddagger}\beta$. We then proceed to form a *new topological space* $\mathbb{J}$ by **gluing** ${}^{\dagger}\mathbb{I}$ to ${}^{\ddagger}\mathbb{I}$ via the *unique isomorphism of topological spaces* ${}^{\dagger}\beta \overset{\sim}{\to} {}^{\ddagger}\alpha$. Thus, ${}^{\dagger}\beta$ and ${}^{\ddagger}\alpha$ are **identified** in $\mathbb{J}$. Let us write $\gamma_{\mathbb{J}}$ for the one-pointed topological space obtained by identifying ${}^{\dagger}\beta$ and ${}^{\ddagger}\alpha$. Thus, $\mathbb{J}$ may be visualized as follows:



(ii) Observe that the gluing operation that gave rise to $\mathbb{J}$ is such that we may regard ${}^{\dagger}\mathbb{I}$ and ${}^{\ddagger}\mathbb{I}$ as *subspaces* ${}^{\dagger}\mathbb{I} \subseteq \mathbb{J}$, ${}^{\ddagger}\mathbb{I} \subseteq \mathbb{J}$ of $\mathbb{J}$. Since each of these subspaces ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$ of $\mathbb{J}$ is naturally isomorphic to $\mathbb{I}$, one may take the point of view, as in the discussion of Example 2.3.1, that these two subspaces are **"redundant"** and hence should be **identified** with one another [say, via the natural isomorphisms of ${}^{\dagger}\mathbb{I}$, ${}^{\dagger}\mathbb{I}$ with $\mathbb{I}$] to form a *new topological space*

$$\mathbb{M} \overset{\text{def}}{=} \mathbb{J}/\langle {}^{\dagger}\mathbb{I} \sim {}^{\ddagger}\mathbb{I} \rangle$$

— where we observe that the natural isomorphisms of ${}^{\dagger}\mathbb{I}$, ${}^{\dagger}\mathbb{I}$ with $\mathbb{I}$ determine a *natural isomorphism* of topological spaces $\mathbb{M} \overset{\sim}{\to} \mathbb{L} = \mathbb{I}/\langle \alpha \sim \beta \rangle$, with the **loop** $\mathbb{L}$ considered in Example 2.3.1. Write $\gamma_{\mathbb{M}} \subseteq \mathbb{M}$ for the image of $\gamma_{\mathbb{J}} \subseteq \mathbb{J}$ in $\mathbb{M}$. Thus, the natural isomorphism $\mathbb{M} \overset{\sim}{\to} \mathbb{L}$ maps $\gamma_{\mathbb{M}}$ isomorphically onto $\gamma_{\mathbb{L}}$. On the other hand, just as in the situation discussed in Example 2.3.1,

> it is *by no means* the case that the fact that ${}^{\dagger}\mathbb{I}$ and ${}^{\ddagger}\mathbb{I}$ are [in fact, naturally] isomorphic as topological spaces implies a sort of **"redundancy"** to the effect that any mathematical argument involving $\mathbb{J}$ is **entirely equivalent** to a corresponding mathematical argument in which ${}^{\dagger}\mathbb{I}$ and ${}^{\ddagger}\mathbb{I}$ are **identified** [say, via the natural isomorphisms of ${}^{\dagger}\mathbb{I}$, ${}^{\dagger}\mathbb{I}$ with $\mathbb{I}$], i.e., in which "$\mathbb{J}$" is *replaced* by "$\mathbb{M}$".

Indeed, for instance, one verifies immediately, just as in the situation of Example 2.3.1, that the fundamental groups of $\mathbb{J}$ and $\mathbb{M}$ are *not isomorphic.* That is to say, just as in the situation of Example 2.3.1, the topological space $\mathbb{J}$ is *structurally/qualitatively very different* from the topological space $\mathbb{M}$.

§2.4. Logical AND "$\wedge$" vs. logical OR "$\vee$"

The essential mathematical content of the *elementary geometric examples* discussed in §2.3 may be reformulated in terms of the *symbolic logical relators* AND "$\wedge$" and OR "$\vee$". This reformulation renders the elementary geometric examples of §2.3 in a form that is even more *directly technically related* to various central aspects of the assertions of the RCS concerning inter-universal Teichmüller theory.

Example 2.4.1: "$\wedge$" vs. "$\vee$" for adjacent oriented line segments.

(i) Recall the situation discussed in Example 2.3.2. Thus, $\mathbb{J} \supseteq {}^{\dagger}\mathbb{I} \supseteq {}^{\dagger}\beta = \gamma_{\mathbb{J}} = {}^{\ddagger}\alpha \subseteq {}^{\ddagger}\mathbb{I} \subseteq \mathbb{J}$, i.e., (AOL1) the following condition **holds**:

$$\left(\gamma_{\mathbb{J}} = {}^{\dagger}\beta \subseteq {}^{\dagger}\mathbb{I}\right) \quad \wedge \quad \left(\gamma_{\mathbb{J}} = {}^{\ddagger}\alpha \subseteq {}^{\ddagger}\mathbb{I}\right).$$

On the other hand, if one **identifies** ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$, then one obtains a topological space $\mathbb{M} \overset{\sim}{\to} \mathbb{L}$, i.e., a **loop**. Here, "$\overset{\sim}{\to}$" denotes the *natural isomorphism* discussed in Example 2.3.2, (ii). Now suppose that we are given a connected subspace

$$\gamma_{\mathbb{I}} \subseteq \mathbb{I}$$

whose image in the quotient $\mathbb{I} \twoheadrightarrow \mathbb{L} = \mathbb{I}/\langle \alpha \sim \beta \rangle$ *coincides* with $\gamma_{\mathbb{L}} \subseteq \mathbb{L}$, i.e., with the image of $\gamma_{\mathbb{J}} \subseteq \mathbb{J}$ via the composite of the quotient $\mathbb{J} \twoheadrightarrow \mathbb{M} = \mathbb{J}/\langle {}^{\dagger}\mathbb{I} \sim {}^{\ddagger}\mathbb{I} \rangle$ with the natural isomorphism $\mathbb{M} \overset{\sim}{\to} \mathbb{L}$. Then *observe* that

(AOL2) the following condition **holds**: $\gamma_{\mathbb{I}} \in \{\alpha, \beta\}$, i.e.,

$$\left(\gamma_{\mathbb{I}} = \beta \subseteq \mathbb{I}\right) \quad \vee \quad \left(\gamma_{\mathbb{I}} = \alpha \subseteq \mathbb{I}\right).$$

Of course,

(AOL3) the essential mathematical content discussed in this condition (AOL2) may be *formally* described as a condition involving the *AND relator* "$\wedge$":

$$\left(\beta \in \{\alpha, \beta\}\right) \quad \wedge \quad \left(\alpha \in \{\alpha, \beta\}\right).$$

But the essential mathematical content of the *OR relator* "$\vee$" statement in (AOL2) remains unchanged.

(ii) On the other hand, [unlike the case with $\gamma_{\mathbb{J}}$!]

(AOL4) the following condition does **not** hold:

$$\left(\gamma_{\mathbb{I}} = \beta \subseteq \mathbb{I}\right) \quad \wedge \quad \left(\gamma_{\mathbb{I}} = \alpha \subseteq \mathbb{I}\right).$$

That is to say, in summary, the operation of **identifying** ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$ — e.g., on the grounds of **"redundancy"** [cf. the discussion of Example 2.3.2] — has the effect of passing from a situation in which

the **AND relator** *"$\wedge$" holds [cf. (AOL1)]*

to a situation in which

the **OR relator** *"$\vee$" holds [cf. (AOL2), (AOL3)],* but
the **AND relator** *"$\wedge$" does* **not** *hold [cf. (AOL4)]!*

(iii) It turns out that this phenomenon — i.e., of an identification of "redundant copies" leading to a passage from the *validity* of an "$\wedge$" relation to the *validity* of an "$\vee$" relation coupled with the *invalidity* of an "$\wedge$" relation — forms a **very precise model** of the situation that arises in the assertions of the RCS concerning inter-universal Teichmüller theory [cf. the discussion of §3.2, §3.4 below].

Example 2.4.2: Differentials on oriented line segments.

(i) In the situation of Example 2.4.1, one way to understand the gap between (AOL1) and (AOL4) — i.e., the *central issue* of whether the *AND relator* "$\wedge$" *holds* or does *not hold* — is to think in terms of the restriction to $\mathbb{I} \subseteq \mathbb{R}$ of the *coordinate function* "x" of Example 2.2.1, (i). Indeed,

(AOD1) one may interpret (AOL4) as the statement that the coordinate functions "x" on the two copies ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$ that constitute $\mathbb{J}$ do **not glue together** to form a single, well-defined $\mathbb{R}$-valued function on $\mathbb{J}$ [that is to say, since it is not clear whether the value of such a function on $\gamma_{\mathbb{J}} \subseteq \mathbb{J}$ should be 0 or 1, i.e., such a function is *not well-defined* on $\gamma_{\mathbb{J}} \subseteq \mathbb{J}$];

(AOD2) on the other hand, (AOL1) may be interpreted as the statement that such a function [i.e., obtained by *gluing together* the coordinate functions "x" on the two copies ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$ that constitute $\mathbb{J}$] **can indeed be defined** if one regards its values as being [not in a *single copy* of $\mathbb{R}$, but rather] in the set ${}^{\dagger,\ddagger}\mathbb{R}$ obtained by gluing together *two distinct copies* ${}^{\dagger}\mathbb{R}$, ${}^{\ddagger}\mathbb{R}$ of $\mathbb{R}$ by *identifying* ${}^{\dagger}1 \in {}^{\dagger}\mathbb{R}$ with ${}^{\ddagger}0 \in {}^{\ddagger}\mathbb{R}$.

(ii) On the other hand, if, instead of considering the coordinate function "x", one considers the *differential* "dx" associated to this coordinate function [cf. the discussion of Example 2.2.1, (i)], then one observes immediately that

(AOD3) the **differentials** "dx" on the two copies ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$ that constitute $\mathbb{J}$ do indeed **glue together** to form a single, well-defined differential on $\mathbb{J}$ that, moreover, is *compatible* with the quotient $\mathbb{J} \twoheadrightarrow \mathbb{M} = \mathbb{J}/\langle {}^{\dagger}\mathbb{I} \sim {}^{\ddagger}\mathbb{I} \rangle$ in the sense that, as is easily verified, it arises as the pull-back, via this quotient map $\mathbb{J} \twoheadrightarrow \mathbb{M}$, of a [smooth] differential on the [smooth manifold constituted by the] loop $\mathbb{M}$.

Note, moreover, that the gluings and compatibility of (AOD3) may be achieved without considering functions or differentials valued in some sort of complicated [i.e., by comparison to $\mathbb{R}$!] set such as ${}^{\dagger,\ddagger}\mathbb{R}$.

(iii) It turns out [cf. the discussion of Example 2.4.1, (iii)] that the phenomenon discussed in (AOD3) is *closely related* to the situation that arises in inter-universal Teichmüller theory [cf. the discussion of §3.2 below].

Example 2.4.3: Representation via subgroup indices of "$\wedge$" vs. "$\vee$".

(i) Let A be an *abelian group* and $B_1, B_2 \subseteq A$ *subgroups* of A such that $B_1 \cap B_2$ has *finite index* in B_1 and B_2. Then one may define a positive rational number, which we call the *index* of B_2 relative to B_1,

$$[B_1 : B_2] \stackrel{\text{def}}{=} [B_1 : B_1 \cap B_2]/[B_2 : B_1 \cap B_2] \in \mathbb{Q}_{>0}.$$

Thus, $[B_1 : B_2] \cdot [B_2 : B_1] = 1$; when $B_2 \subseteq B_1$, this notion of index coincides with the usual notion of the index of B_2 in B_1.

(ii) Let n be a positive integer ≥ 2. Consider the *diagram of group homomorphisms*

$$\begin{tikzcd} G_1 \arrow[r, "n\cdot"] & G_2 \arrow[r, "n\cdot"] & G_3 \end{tikzcd}$$

— where, for $i = 1, 2, 3$, G_i denotes a copy of [the additive group of rational integers] $\mathbb{Z}$, and the arrows are given by *multiplication* by n. For $i = 1, 2, 3$, write $G_i^{\mathbb{Q}} \overset{\text{def}}{=} G_i \otimes_{\mathbb{Z}} \mathbb{Q}$ for the tensor product of G_i over $\mathbb{Z}$ with $\mathbb{Q}$. Then observe that this diagram induces a *diagram of group isomorphisms*

$$G_1^{\mathbb{Q}} \overset{\sim}{\rightarrow} G_2^{\mathbb{Q}} \overset{\sim}{\rightarrow} G_3^{\mathbb{Q}}$$

— i.e., in which the arrows are *isomorphisms*. Let us use these isomorphisms to identify the groups $G_i^{\mathbb{Q}}$, for $i = 1, 2, 3$, and denote the resulting group by $G_*^{\mathbb{Q}}$.

(iii) Observe that the *first diagram* of (ii) is *structurally reminiscent* of the object $\mathbb{J}$ discussed in Examples 2.3.2, 2.4.1, 2.4.2, i.e., if one regards

- the *first arrow* of the first diagram of (ii) as corresponding to ${}^{\dagger}\mathbb{I}$,
- the *second arrow* of the first diagram of (ii) as corresponding to ${}^{\ddagger}\mathbb{I}$, and
- G_1, G_2, and G_3 as corresponding to ${}^{\dagger}\alpha$, ${}^{\dagger}\beta = {}^{\ddagger}\alpha$, and ${}^{\ddagger}\beta$, respectively.

Here, we observe that G_2 appears **simultaneously** as the *codomain* of the arrow $G_1 \xrightarrow{n\cdot} G_2$ **AND** [cf. (AOL1)!] as the *domain* of the arrow $G_2 \xrightarrow{n\cdot} G_3$. Moreover, we may consider *indices* of G_1, G_2, and G_3 as *subgroups* of $G_*^{\mathbb{Q}}$

$$[G_2 : G_1] = [G_1 : G_2]^{-1} = n; \quad [G_3 : G_2] = [G_2 : G_3]^{-1} = n;$$
$$[G_3 : G_1] = [G_1 : G_3]^{-1} = n^2$$

in a **consistent** fashion, i.e., in a fashion that does **not** give rise to any **contradictions**.

(iv) On the other hand, suppose that we *delete* the *"distinct labels"* G_1, G_2, G_3 from the copies of $\mathbb{Z}$ considered in the *first diagram* of (ii). This yields a *diagram*

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "n\cdot"] & \mathbb{Z} \arrow[r, "n\cdot"] & \mathbb{Z} \end{tikzcd}$$

in which the *second arrow* may be regarded as a *copy* of the *first arrow*. This situation might motivate some observers to conclude that these two arrows are **"redundant"** and hence should be **identified** with one another — cf. the discussion of the quotient $\mathbb{J} \twoheadrightarrow \mathbb{M}$ in Example 2.3.2, (ii) — to form a *diagram*

$$\overset{n\cdot}{\curvearrowright} \mathbb{Z}$$

consisting of a *single copy* of $\mathbb{Z}$ and the *endomorphism* of this single copy of $\mathbb{Z}$ given by *multiplication* by n. At first glance, this operation of **identification** may appear to give rise to various **"contradictions"** in the computation of the *index*, i.e., such as

$$1 = [G_1 : G_1] = [\mathbb{Z} : \mathbb{Z}] = [G_2 : G_1] = n \geq 2$$

and so on. In fact, however, if one takes into account the *OR relator* "$\vee$" [but not the *AND relator* "$\wedge$"!] relations that one obtains upon executing the identification operation in question [cf. (AOL2), (AOL4)!], then one concludes that [after executing the identification operation in question!] each of the indices $[G_i : G_j]$, for $i, j \in \{1, 2, 3\}$, *may only be computed up to multiplication by an integral power of* n, i.e., that

> each index $[G_i : G_j]$, for $i, j \in \{1, 2, 3\}$, is only well-defined as **"some indeterminate element"** of $n^{\mathbb{Z}} \overset{\text{def}}{=} \{n^m \mid m \in \mathbb{Z}\} \subseteq \mathbb{Q}_{>0}$.

In particular, in fact there is **no contradiction**.

Example 2.4.4: Logical "$\wedge/\vee$" vs. "narrative $\wedge/\vee$". Consider the following argument concerning a *natural number* $x \in \{1, 3\}$, i.e., a natural number for which it holds that $(x = 1) \ \vee \ (x = 3)$:

(Nar1) Suppose that $x = 3$. Then it follows that $x = 3 > 2$. That is to say, we *conclude* that $x > 2$.

(Nar2) Since $(x = 1) \ \vee \ (x = 3)$, we may consider the case $x = 1$. Then, by applying the conclusion of (Nar1), we conclude that $1 = x > 2$, i.e., that $1 > 2$ — a *contradiction*!

Of course, this argument is *completely fallacious*! On the other hand, it yields a readily understood concrete example of the *absurdity* that arises when, as is in effect done in (Nar2), *logical OR "$\vee$"* is *confused* with *logical AND "$\wedge$"*! In various contexts, this sort of confusion can arise from the *ambiguity of various narrative expressions* that appear in the discussion of a mathematical argument. This sort of ambiguity can lead to a situation in which

> a **"narrative AND $\wedge$"** — i.e., the fact that in a particular narrative exposition of an argument, one performs **both** the task of considering the case "$x = 3$" [cf. (Nar1)] **and** the task of considering the case "$x = 1$" [cf. (Nar2)] — is mistakenly construed as a **logical AND "$\wedge$"**.

In a similar vein, one may consider situations in which the roles played by "$\wedge$" and "$\vee$" are *reversed*, i.e., in which a **"narrative OR $\vee$"** — i.e., the fact that in a particular narrative exposition of an argument, one's attention is concentrated **either** on the task of considering one situation **or** on the task of considering another situation — is mistakenly construed as a **logical OR "$\vee$"**. Indeed, it appears that one *fundamental cause*, in the context of the *essential logical structure* of inter-universal Teichmüller theory [cf. the discussion of Example 2.4.5 below!], of the confusion on the part of some mathematicians between logical AND "$\wedge$" and logical OR "$\vee$" lies precisely in this sort of confusion between **"narrative $\wedge/\vee$"** and **logical "$\wedge/\vee$"**.

Example 2.4.5: Numerical representation of "$\wedge$" vs. "$\vee$".

(i) A slightly more sophisticated numerical representation of the difference between "$\wedge$" and "$\vee$" — which in fact *mirrors the* **essential logical structure** *of inter-universal Teichmüller theory in a very direct fashion* — may be given as follows [cf. [Alien], Example 3.11.4]. Indeed, the *essential logical flow* of inter-universal Teichmüller theory may be summarized as follows:

- one starts with the definition of an object called the **Θ-link**;
- one then constructs a complicated apparatus that is referred to as the **multiradial representation of the Θ-pilot** [cf. [IUTchIII], Theorem 3.11];
- finally, one derives a **final numerical estimate** [cf. [IUTchIII], Corollary 3.12] in an essentially straightforward fashion from the multiradial representation of the Θ-pilot.

(ii) An elementary model of this essential logical flow may be given by means of *real numbers* $A, B \in \mathbb{R}_{>0}$ and $\epsilon, N \in \mathbb{R}$ such that $0 \leq \epsilon \leq 1$ in the following way:

- **Θ-link**:

$$\left(N \overset{\mathrm{def}}{=} -2B\right) \quad \wedge \quad \left(N \overset{\mathrm{def}}{=} -A\right);$$

- **multiradial representation of the Θ-pilot**:

$$\left(N = -2A + \epsilon\right) \quad \wedge \quad \left(N = -A\right);$$

- **final numerical estimate**:

$$-2A + \epsilon = -A, \text{ hence } A = \epsilon, \text{ i.e., } A \leq 1.$$

Thus, the definition of the Θ-*link* and the construction of the *multiradial representation of the* Θ-*pilot* are *meaningful/nontrivial* precisely on account of the validity of the **AND relator** "$\wedge$", which is rendered possible, in the definition of the Θ-link, precisely by allowing the real numbers A, B to be [*a priori*] **distinct** real numbers — cf. (AOL1) vs. (AOL4), where we think in terms of the *correspondences*

$$\begin{array}{ccccccccc} B & \longleftrightarrow & {}^{\dagger}\mathbb{I}, & N & \longleftrightarrow & \gamma_{\mathbb{J}}, & A & \longleftrightarrow & {}^{\ddagger}\mathbb{I} \\ -2B & \longleftrightarrow & {}^{\dagger}\beta, & -A & \longleftrightarrow & {}^{\ddagger}\alpha, & -2A & \longleftrightarrow & {}^{\ddagger}\beta. \end{array}$$

The passage from the *multiradial representation of the* Θ-*pilot* to the *final numerical estimate* is then *immediate/straightforward/logically transparent*.

(iii) By contrast, if, in the elementary numerical model of (ii), one *replaces* "$\wedge$" by "$\vee$", then our elementary numerical model of the logical structure of inter-universal Teichmüller theory takes the following form:

- **"$\vee$" version of Θ-link**:

$$\left(N \overset{\mathrm{def}}{=} -2B\right) \quad \vee \quad \left(N \overset{\mathrm{def}}{=} -A\right) \qquad [\text{cf. } \left(N \overset{\mathrm{def}}{=} -2A\right) \quad \vee \quad \left(N \overset{\mathrm{def}}{=} -A\right)];$$

- **"$\vee$" version of multiradial representation of the Θ-pilot**:

$$\left(N = -2A + \epsilon\right) \quad \vee \quad \left(N = -A\right);$$

- **final numerical estimate**:

$$-2A + \epsilon = -A, \text{ hence } A = \epsilon, \text{ i.e., } A \leq 1.$$

That is to say, the use of *distinct* real numbers A, B in the definition of the "$\vee$" version of Θ-link seems **entirely superfluous** [cf. (AOL2), relative to the correspondences discussed in (ii)]. This motivates one to **identify** A and B — i.e., to suppose *"for the sake of simplicity"* that $A = B$ — which then has the effect of rendering the definition of the original "$\wedge$" version of the Θ-link *invalid/self-contradictory* [cf. (AOL4), relative to the correspondences discussed in (ii)]. Once one *identifies* A and B, i.e., once one supposes "for the sake of simplicity" that $A = B$, the passage from the "$\vee$" version of Θ-link to the resulting "$\vee$" version of the *multiradial representation of the* Θ*-pilot* then seems **entirely meaningless/devoid of any interesting content**. The passage from the resulting meaningless "$\vee$" version of the *multiradial representation of the* Θ*-pilot* to the *final numerical estimate* then seems **abrupt/mysterious/entirely unjustified**, i.e., put another way, looks as if

> one **erroneously replaced** the "$\vee$" in the meaningless "$\vee$" version of the multiradial representation of the Θ-pilot by an "$\wedge$" **without any mathematical justification whatsoever**.

It is precisely this **pernicious chain of misunderstandings** emanating from the **"redundancy"** assertions of the RCS that has given rise to a substantial amount of unnecessary confusion concerning inter-universal Teichmüller theory.

(iv) Before proceeding, we observe that the sort of **confusion** discussed in (iii) between "$\wedge$" and "$\vee$" can occur as the result of any of the following phenomena:

(AOC1) a confusion between **"narrative $\wedge/\vee$"** and **logical "$\wedge/\vee$"**, as discussed in Example 2.4.4;

(AOC2) thinking in terms of the **"fake $\wedge$"** of (AOL3), i.e., which, though formulated as a logical AND "$\wedge$" relation, is in fact, substantively speaking, a *logical OR "$\vee$"* relation;

(AOC3) the *symptom (Syp2)* discussed in §3.6 below, i.e., a desire to see the "proof" of some sort of **commutative diagram** or **"compatibility property"** to the effect that taking *log-volumes of pilot objects* in the domain and codomain of the Θ-link yields the *same real number*;

(AOC4) a **fundamental misunderstanding** — which is often closely intertwined with the symptom (Syp2) discussed in (AOC3) — of the meaning of the crucial **closed loop** of §3.10, (Stp7), (Stp8), below [cf. §3.10, (Stp7), (Stp8), as well as the following discussion].

(v) Let us refer to the "$\wedge$" version of inter-universal Teichmüller theory discussed in (ii) — i.e., the *original version* of inter-universal Teichmüller theory, in which one interprets the **Θ-link** as a **logical AND "$\wedge$"** relation — as **AND-IUT**. Thus,

AND-IUT = IUT is the *original version* of inter-universal Teichmüller theory.

Let us refer to the "$\vee$" version of inter-universal Teichmüller theory discussed in (iii) — i.e., the version of inter-universal Teichmüller theory that arises if one [*mistakenly*!] interprets the **Θ-link** as a **logical OR "$\vee$"** relation — as **OR-IUT**. As discussed in (iii), in OR-IUT, one is motivated to implement the RCS-identifications of RCS-redundant copies of objects — i.e., in the language of (iii), to *"identify A and B"* — and hence to conclude that **OR-IUT $\Longrightarrow$ RCS-IUT**, where we recall that "RCS-IUT" refers to the version of inter-universal Teichmüller theory obtained by implementing the RCS-identifications of RCS-redundant copies of objects [cf. the discussion of §1.2]. On the other hand, it is not difficult to see that in RCS-IUT, one is forced to work with a *(NeuORInd) indeterminacy* [cf. the discussion at the end of §3.4 below, as well as the discussion of (ΘORInd) in §3.11 below], i.e., to interpret the **Θ-link** as a **logical XOR "$\dot{\vee}$"** relation [that is to say, a **logical OR "$\vee$"** relation such that the corresponding **logical AND "$\wedge$"** relation *cannot hold* — cf. the discussion of (iii)]. In particular, we conclude that **RCS-IUT $\Longrightarrow$ XOR-IUT $\Longrightarrow$ OR-IUT** [where the second " $\Longrightarrow$ " is a consequence of well-known general properties of Boolean operators], i.e., in summary:

(XOR/RCS) we have *equivalences* **XOR-IUT $\Longleftrightarrow$ OR-IUT $\Longleftrightarrow$ RCS-IUT**.

In the following, I shall refer to the *school of thought* [i.e., in the sense of a "collection of closely interrelated ideas"] surrounding OR-IUT as **ORS**, i.e., the "OR school [of thought]", and to the *school of thought* surrounding XOR-IUT as **XORS**, i.e., the "XOR school [of thought]". Thus, **XORS = ORS = RCS**.

(vi) On the other hand, one may also consider yet another version of inter-universal Teichmüller theory, also motivated by the discussion of (iii), which we refer to as **EssOR-IUT**, i.e., "essentially OR IUT". This is the version of inter-universal Teichmüller theory in which one accepts, at the level of formal definitions, the **logical AND "$\wedge$"** version of the **Θ-link** as in (ii), i.e., *without identifying A and B*, but [for some unexplained reason!] one then *arbitrarily shifts*, when considering the **multiradial representation of the Θ-pilot**, to the **logical OR "$\vee$"** interpretation of the multiradial representation of the Θ-pilot, i.e., as in (iii). That is to say, as the name "EssOR-IUT" suggests, the fundamental *logical AND "$\wedge$"* property of the *Θ-link* is *never actually used in any sort of meaningful way* in EssOR-IUT. In particular,

(EssOR/RCS) although, at a purely *formal* level, **EssOR-IUT** *rejects* **RCS-IUT**, the *essential logical structure* of EssOR-IUT still nevertheless gives rise to the **abrupt/mysterious/entirely unjustified** transition discussed in (iii) to the *final numerical estimate*.

It appears that the *"arbitrarily shift"* referred to above is often precipitated by the various phenomena discussed in (iv) [cf., especially, (AOC3), (AOC4)]. In the following, I shall refer to the *school of thought* [i.e., in the sense of a "collection of closely interrelated ideas"] surrounding EssOR-IUT as **EssORS**, i.e., the "essentially OR school [of thought]".

(vii) In general, at the level of formalities of *Boolean operators*, "$\wedge \Rightarrow \vee$", but "$\vee \not\Rightarrow \wedge$". In particular, in the context of the transition to the *final numerical estimate* of inter-universal Teichmüller theory,

$(\vee \not\Rightarrow \wedge)$ it appears entirely *hopeless/unrealistic* to pass from the "$\vee$" version of the *multiradial representation of the* Θ*-pilot* to the "$\wedge$" version of the multiradial representation of the Θ-pilot.

This is precisely the *"abrupt/mysterious/entirely unjustified"* transition [to the *final numerical estimate*] discussed in (iii).

(viii) The discussion of (v), (vi), (vii) may be summarized as follows [cf. also the discussion of §1.2, §1.3]:

- The **fundamental misunderstanding** on the part of adherents of the RCS = ORS = XORS to the effect that OR-IUT or XOR-IUT is indeed the content of AND-IUT = IUT leads to the *mistaken interpretation* of the assertion (XOR/RCS) as an *equivalence between AND-IUT = IUT and RCS-IUT.*

- The **fundamental misunderstanding** on the part of adherents of the RCS = ORS = XORS to the effect that OR-IUT or XOR-IUT is indeed the content of AND-IUT = IUT leads to the *mistaken interpretation* of the assertion $(\vee \not\Rightarrow \wedge)$ as a *logical flaw* in AND-IUT = IUT.

- The **fundamental misunderstanding** on the part of adherents of the EssORS to the effect that EssOR-IUT is indeed the content of AND-IUT = IUT leads *either* to the *mistaken interpretation* of the assertion $(\vee \not\Rightarrow \wedge)$ as a *logical flaw* in AND-IUT = IUT *or* to the *mistaken interpretation* of the assertion $(\vee \not\Rightarrow \wedge)$ as an indication the existence of some sort of *infinitely complicated and mysterious argument* — i.e., for concluding that "$\vee \Rightarrow \wedge$"! — in inter-universal Teichmüller theory that requires years of concerted effort to understand. Thus, as the descriptive *"essential"* suggests, there is in fact, from the point of view of the *essential logical structure* under consideration, *very little difference* between EssORS and XORS = ORS = RCS or between EssOR-IUT and XOR-IUT $\iff$ OR-IUT $\iff$ RCS-IUT.

- In fact, the **correct interpretation** of the assertion $(\vee \not\Rightarrow \wedge)$ consists of the conclusion that *neither XOR-IUT* $\iff$ *OR-IUT* $\iff$ *RCS-IUT nor EssOR-IUT has any direct logical relationship to AND-IUT = IUT.*

Here, we observe that the above analysis is in some sense *remarkable* in that it makes explicit the fact that, if one **forgets** the **arbitrary label "inter-universal Teichmüller theory"** placed on XOR-IUT or OR-IUT or EssOR-IUT by adherents of the RCS = ORS = XORS or the EssORS, then [perhaps somewhat surprisingly!]

(MthVl) there is in fact **no disagreement** among any of the parties involved with regard to the **mathematical validity** of the mathematical assertions (XOR/RCS) and $(\vee \not\Rightarrow \wedge)$.

Indeed, this state of affairs may be understood as in some sense highlighting the *essentially social/political/psychological, i.e., in summary, non-mathematical* nature of the *entirely unnecessary confusion* that has arisen concerning inter-universal Teichmüller theory [cf. the discussion of §1.8, §1.11]. The above observations are summarized in the **"dictionary of assertions"** given below.

Assertions of various schools of thought	*Actual mathematical content*
RCS = ORS = XORS: "IUT $\Leftrightarrow$ RCS-IUT"	"XOR-IUT $\Leftrightarrow$ OR-IUT $\Leftrightarrow$ RCS-IUT"
RCS = ORS = XORS: "IUT is logically flawed."	"$\vee \not\Rightarrow \wedge$", which implies that "(AND-IUT =) IUT $\not\Rightarrow$ RCS-IUT", "(AND-IUT =) IUT $\not\Rightarrow$ OR-IUT", "(AND-IUT =) IUT $\not\Rightarrow$ XOR-IUT"
EssORS: *either* "IUT is logically flawed." *or* "The logical structure of IUT is infinitely complicated/mysterious."	"$\vee \not\Rightarrow \wedge$", which implies that "(AND-IUT =) IUT $\not\Rightarrow$ EssOR-IUT"

Example 2.4.6: Carry operations in arithmetic, geometry, and Boolean logic.

(i) Observe that if, in the situation of Example 2.4.3, (ii), one focuses one's attention on the subset $D_{4-i} \subseteq G_i$ in the copy of $\mathbb{Z}$ denoted by G_i, where $i = 1, 2, 3$, corresponding to $\{0, 1, 2, \ldots, n-1\} \subseteq \mathbb{Z}$, then the situation considered in Example 2.4.3, (ii), closely resembles the situation that arises in *elementary arithmetic computations* — such as *addition* and *multiplication* — involving **base n expansions of natural numbers**. That is to say, one may think of

- D_1 as the *first digit*, i.e., when $n = 10$, the **"ones digit"**,
- D_2 as the *second digit*, i.e., when $n = 10$, the **"tens digit"**, and
- D_3 as the *third digit*, i.e., when $n = 10$, the **"hundreds digit"**

of such an expansion. When performing such elementary arithmetic computations — such as addition and multiplication — involving base n expansions of natural numbers, recall that it is of *fundamental importance* to take into account the various **carry operations** that occur. In particular, we observe that

> the use of **distinct labels** for **distinct digits** plays a *fundamental role* in elementary arithmetic computations involving *base n expansions of natural numbers*

— cf. the *distinct labels* "G_1", "G_2", "G_3" in the discussion of Example 2.4.3, (ii), (iii). This situation is reminiscent of the important role played by the **distinct labels** "A", "B" in the **"Θ-link"** of Example 2.4.5, (ii). Note, morever, that

> **deletion/confusion** of these **distinct labels** for **distinct digits** has the effect of **completely invalidating**, at least in the usual "strict sense", elementary arithmetic computations involving *base n expansions of natural numbers*

— cf. the situation considered in Example 2.4.3, (iv). On the other hand, if one restricts one's attention to a **specific computational algorithm** [involving, say, addition and multiplication operations], then in fact it is often possible — i.e., depending on the content of the specific computational algorithm under consideration — to obtain

> **estimates** to the effect that applying the algorithm either *with* or *without* the use of *distinct labels* for *distinct digits* in the base n expansions of the natural numbers involved in fact yields the **same result**, up to some **explicitly bounded discrepancy**.

[For instance, when $n = 10$, any algorithm that only involves *addition* and *multiplication* operations yields the *same result modulo* 9 $(= 10 - 1)$, regardless of whether or not one uses *distinct labels* for *distinct digits* in decimal expansions of natural numbers.] Such estimates are reminiscent of the **"multiradial representation"** of Example 2.4.5, (ii).

(ii) Observe that the discussion of

· **adjacent oriented line segments** "$^{\dagger}\mathbb{I}$", "$^{\ddagger}\mathbb{I}$" and
· **oriented loops** "$\mathbb{L}$", "$\mathbb{M}$"

in Examples 2.3.1, 2.3.2, 2.4.1, 2.4.2 [cf. also the discussion of Examples 2.4.3, (iii); 2.4.5, (ii)] may be regarded as a sort of **limiting case** of the discussion of **base n expansions of natural numbers** in (i) above, i.e., if one

· considers the *real numbers* obtained by *dividing* the natural numbers $\leq 2n$ in the discussion of (i) above *by* n and then

· passes to the *limit* $n \to +\infty$.

That is to say, in summary,

> the **adjacency** of the *oriented line segments* "$^{\dagger}\mathbb{I}$", "$^{\ddagger}\mathbb{I}$" may be understood as a sort of **continuous, geometric representation** of the **carry operation** that appears in elementary arithmetic computations involving *base n expansions of natural numbers.*

(iii) From the point of view of discussions of the **logical structure** of mathematical arguments represented in terms of **Boolean operators** such as *logical AND "$\wedge$"* and *logical OR "$\vee$"*, it is of interest to consider the discussion of (i) above in the **binary** case, i.e., the case $n = 2$. We begin our discussion of the binary case by recalling the following well-known facts:

· **multiplication** in the *field* $\mathbb{F}_2 = \{0, 1\}$ may be regarded as corresponding to the **Boolean operator AND "$\wedge$"**;

- **addition** in the *field* $\mathbb{F}_2 = \{0, 1\}$ may be regarded as corresponding to the **Boolean operator XOR** [i.e., *"exclusive OR"*], which we denote by "$\dot{\vee}$";

- **"carry-addition"** in the *truncated ring of Witt vectors* $\mathbb{F}_2 \times \mathbb{F}_2$ — i.e., addition of two elements of $\mathbb{F}_2 \overset{\sim}{\to} \{0\} \times \mathbb{F}_2 \subseteq \mathbb{F}_2 \times \mathbb{F}_2$, regarded as *Teichmüller representatives* in the truncated ring of Witt vectors $\mathbb{Z}/4\mathbb{Z}$, that is to say, an addition operation in which one allows for the **carry operation** [cf. the discussion of (i)!] to the *first factor* of $\mathbb{F}_2 \times \mathbb{F}_2$ — may be regarded as corresponding to an operator that we shall refer to as the **"COR", or "carry-OR", operator** and denote by "$\ddot{\vee}$"; thus, we have

$$\ddot{\vee} = (\wedge, \dot{\vee})$$

[so $0 \ddot{\vee} 0 \ = \ (0, 0)$; $1 \ddot{\vee} 0 \ = \ 0 \ddot{\vee} 1 \ = \ (0, 1)$; $1 \ddot{\vee} 1 \ = \ (1, 0)$].

These well-known facts case may be summarized as follows:

($\ddot{\vee} = \wedge \dot{\vee}$) **Conventional mixed-characteristic/"carry" addition** in $\mathbb{Z}$ considered modulo 4 — i.e., "$\ddot{\vee}$" — may be described in terms of the *"splitting"* of the natural surjection $\mathbb{Z} \twoheadrightarrow \mathbb{F}_2$ determined by the *"Teichmüller representatives"* $0, 1 \in \mathbb{Z}$ via the equation

$$\ddot{\vee} \ = \ (\wedge, \dot{\vee})$$

— i.e., which exhibits "$\ddot{\vee}$" as an operation obtained by *"stacking"* **multiplication** *"$\wedge$" in $\mathbb{F}_2$ on top of* **addition** *"$\dot{\vee}$" in $\mathbb{F}_2$*. Here, we note that this splitting via Teichmüller representatives $0, 1 \in \mathbb{Z}$ is *compatible* with the **multiplicative** structures in $\mathbb{Z}$ and $\mathbb{F}_2$, but **not** with the **additive** structures in $\mathbb{Z}$ and $\mathbb{F}_2$. Put another way, one may think of the ring structures of $\mathbb{Z}$ and $\mathbb{F}_2$ as structures that share a *common multiplicative structure* [cf. "$\wedge$"!], but do *not* share a *common additive structure* [cf. "$\dot{\vee}$"!].

These observations will be of *fundamental importance* in the theory developed in §3 [cf., especially, the discussion at the beginning of §3.10].

(iv) Some readers may object to the comparisons and analogies between inter-universal Teichmüller theory and the **"mathematics of carry operations"** discussed in (i), (ii), and (iii) as being inappropriate on the grounds that this mathematics of carry operations is much too *"trivial"* to be of any substantive interest. On the other hand, we observe that the mathematics of carry operations discussed in (i), (ii), and (iii) is *intimately intertwined* with numerous important developments in the *history of mathematics.* With regard to the content of (i), we recall that the use of **place-value decimal numerals**, i.e., that make use of notation for *zero*, appears to date back to **Indian** texts and inscriptions from the 7-th to 9-th centuries AD. Such numerals also reached the **Arab** world during this period, but this **Hindu-Arabic numeral system** apparently only became widely used in **Europe** during the late middle ages, between the 13-th and 15-th centuries. In this context, one noteworthy development was the book *Liber Abaci* published by the Italian mathematician *Fibonacci* in 1202, which promoted the use of the Hindu-Arabic numeral system in Europe. Here, it is useful to recall that, by comparison to earlier numeral systems, such as the *Greco-Roman* and *Babylonian* systems, *place-value decimal numerals* not only facilitate elementary arithmetic computations — i.e., via the systematic use of **carry operations**, as discussed in (i)! — but also

> make it possible to express **all** — hence, in particular, *infinitely many* — **natural numbers** by means of **finitely many symbols**

— i.e., unlike earlier numerical systems, in which only *finitely many natural numbers* could be expressed using finitely many symbols. This *revolutionary importance* of the development of place-value decimal numerals in India was recognized, for instance, in writings of the 18-th century French mathematician *Laplace.* In this context, it is also of interest to observe that the discussion in (ii) of the interpretation of the discussion of (i) in terms of *line segments* is reminiscent of the discussion of the *"Euclidean algorithm"* in *Euclid's Elements*, in which numbers are often represented as *lengths of line segments.* Finally, we recall that the Boolean aspects discussed in (iii) played an important role in the [well-known!] development of *modern digital computers* in the 20-th century.

The *gluings* of *adjacent line segments* discussed in Examples 2.3.2, 2.4.1, 2.4.2 may in some sense be regarded as a sort of

> **optimized elementary geometric/combinatorial representation** of the **essential logical "$\wedge/\vee$" structure** surrounding a **gluing**

in a fashion that is *qualitatively entirely structually similar* to the gluings that occur in inter-universal Teichmüller theory, which will be discussed in more detail in §3 below. The somewhat more *numerical/arithmetic* situations discussed in Examples 2.4.3, 2.4.5, 2.4.6 may also be regarded as *entirely elementary* representations of this *essential logical "$\wedge/\vee$" structure* surrounding a *gluing.* On the other hand, the *gluing* operation that occurs in the standard construction of the **projective line**, while somewhat less elementary than the previously mentioned examples, also constitutes an *important* — and, moreover, still *relatively elementary*! — representation of this *essential logical "$\wedge/\vee$" structure* surrounding a *gluing.* Moreover, this example of the projective line discussed in Example 2.4.7 is *more directly related to* **scheme-theoretic arithmetic geometry** than the previously mentioned examples and helps to motivate the subsequent **ring-/monoid-theoretic** Example 2.4.8, which may literally be regarded, i.e., in a much more *rigorous, technical sense*, as a sort of miniature qualitative model — that is to say, so to speak, a sort of *"preview"* — of the *gluing* constituted by the **Θ-link** of inter-universal Teichmüller theory.

Example 2.4.7: The projective line as a gluing of ring schemes along a multiplicative group scheme. In the following discussion, we take k to be a *field* and $q \in k$ to be an element such that $q^3 \neq q$ [i.e., $q \notin \{0, 1, -1\}$]. Write $k^\times \overset{\mathrm{def}}{=} k \backslash \{0\}$, $\mathbb{A}^1$ for the *affine line* $\mathrm{Spec}(k[T])$ over k, $\mathbb{G}_\mathrm{m}$ for the open subscheme $\mathrm{Spec}(k[T, T^{-1}])$ of $\mathbb{A}^1$ obtained by removing the origin. Thus, the standard coordinate T on $\mathbb{A}^1$, $\mathbb{G}_\mathrm{m}$ determines *natural bijections* $\mathbb{A}^1(k) \xrightarrow{\sim} k$, $\mathbb{G}_\mathrm{m}(k) \xrightarrow{\sim} k^\times$ of the respective sets of k-rational points of $\mathbb{A}^1$, $\mathbb{G}_\mathrm{m}$ with corresponding subsets of k. Also, we recall that $\mathbb{A}^1$ is equipped with a well-known natural structure of **ring scheme** over k, while $\mathbb{G}_\mathrm{m}$ is equipped with a well-known natural structure of [**multiplicative**] **group scheme** over k.

(i) Write ${}^{\dagger}\mathbb{A}^1$, ${}^{\ddagger}\mathbb{A}^1$ for the *k-ring schemes* given by *copies* of $\mathbb{A}^1$ equipped with the respective **labels** "†", "‡". We regard ${}^{\dagger}\mathbb{A}^1$ as being further equipped with the k-rational point ${}^{\dagger}q^{-1} \in {}^{\dagger}\mathbb{A}^1(k)$ ($\xrightarrow{\sim} k$) corresponding to the multiplicative inverse of the element $q \in k$ and ${}^{\ddagger}\mathbb{A}^1$ as being further equipped with the k-rational point ${}^{\ddagger}q \in {}^{\ddagger}\mathbb{A}^1(k)$ ($\xrightarrow{\sim} k$) corresponding to the element $q \in k$. Similarly, we write ${}^{\dagger}\mathbb{G}_{\mathrm{m}}$, ${}^{\ddagger}\mathbb{G}_{\mathrm{m}}$ for the *[multiplicative] k-group schemes* given by *copies* of $\mathbb{G}_{\mathrm{m}}$, equipped with the respective **labels** "†", "‡". Thus, ${}^{\dagger}q^{-1} \in {}^{\dagger}\mathbb{G}_{\mathrm{m}}(k)$ ($\subseteq {}^{\dagger}\mathbb{A}^1(k)$), ${}^{\ddagger}q \in {}^{\ddagger}\mathbb{G}_{\mathrm{m}}(k)$ ($\subseteq {}^{\ddagger}\mathbb{A}^1(k)$).

(ii) Relative to the notation of (i), we observe that

(ii-a) there exists a **unique isomorphism** of ***k*-ring schemes** ${}^{\dagger}\mathbb{A}^1 \xrightarrow{\sim} {}^{\ddagger}\mathbb{A}^1$, but that

(ii-b) the *pairs* $({}^{\dagger}\mathbb{A}^1, {}^{\dagger}q^{-1})$ and $({}^{\ddagger}\mathbb{A}^1, {}^{\ddagger}q)$ are **not isomorphic**, i.e., as pairs consisting of a ***k*-ring scheme** equipped with a ***k*-rational point** [cf. our assumption that $q^3 \neq q$].

By contrast,

(ii-c) there exists a **unique isomorphism** of *pairs* $({}^{\dagger}\mathbb{G}_{\mathrm{m}}, {}^{\dagger}q^{-1}) \xrightarrow{\sim} ({}^{\ddagger}\mathbb{G}_{\mathrm{m}}, {}^{\ddagger}q)$, i.e., of pairs consisting of a [**multiplicative**] ***k*-group scheme** equipped with a ***k*-rational point**.

Here, we observe that the isomorphism $({}^{\dagger}\mathbb{G}_{\mathrm{m}}, {}^{\dagger}q^{-1}) \xrightarrow{\sim} ({}^{\ddagger}\mathbb{G}_{\mathrm{m}}, {}^{\ddagger}q)$ of (ii-c) does **not extend** [cf. (ii-b)!] to an isomorphism $({}^{\dagger}\mathbb{A}^1, {}^{\dagger}q^{-1}) \xrightarrow{\sim} ({}^{\ddagger}\mathbb{A}^1, {}^{\ddagger}q)$. In particular,

(ii-d) the isomorphism of [**multiplicative**] ***k*-group schemes** ${}^{\dagger}\mathbb{G}_{\mathrm{m}} \xrightarrow{\sim} {}^{\ddagger}\mathbb{G}_{\mathrm{m}}$ is **not compatible** with the ***k*-ring scheme structures** of ${}^{\dagger}\mathbb{A}^1$ ($\supseteq {}^{\dagger}\mathbb{G}_{\mathrm{m}}$), ${}^{\ddagger}\mathbb{A}^1$ ($\supseteq {}^{\ddagger}\mathbb{G}_{\mathrm{m}}$).

Next, we observe that

(ii-e) the *standard construction of the* **projective line** may be regarded as the result of **gluing** $({}^{\dagger}\mathbb{A}^1, {}^{\dagger}q^{-1})$ to $({}^{\ddagger}\mathbb{A}^1, {}^{\ddagger}q)$ along the isomorphism

$$({}^{\dagger}\mathbb{G}_{\mathrm{m}}, {}^{\dagger}q^{-1}) \xrightarrow{\sim} ({}^{\ddagger}\mathbb{G}_{\mathrm{m}}, {}^{\ddagger}q)$$

of (ii-c); thus, *relative to this gluing*, ${}^{\dagger}\mathbb{G}_{\mathrm{m}} \xrightarrow{\sim} {}^{\ddagger}\mathbb{G}_{\mathrm{m}}$ may be regarded **simultaneously** as an open subscheme of ${}^{\dagger}\mathbb{A}^1$ **AND** [cf. "$\wedge$"!] as an open subscheme of ${}^{\ddagger}\mathbb{A}^1$.

In particular, (ii-d), (ii-e) may be summarized as follows:

> the standard construction of the **projective line** may be regarded as a **gluing** of two **ring schemes** along an **isomorphism** of **multiplicative group schemes** that is **not compatible** with the **ring scheme** structures on either side of the gluing.

Moreover, we note that, *relative to this gluing*,

(ii-f) the notion of a **regular function** on ${}^{\dagger}\mathbb{A}^1$ **cannot be expressed directly** in terms of the notion of a **regular function** on ${}^{\ddagger}\mathbb{A}^1$, whereas

(ii-g) the notion of a **rational function** on ${}^{\dagger}\mathbb{A}^1$ **can be expressed directly** in terms of — i.e., in essence, *coincides*, relative to the above gluing, with — the notion of a **rational function** on ${}^{\ddagger}\mathbb{A}^1$.

Finally, we observe that

(ii-h) if, in the gluing of (ii-e), one **arbitrarily deletes** the **distinct labels** “†”, “‡”, then the resulting *“gluing without labels”* amounts to a gluing of a **single copy** of $\mathbb{A}^1$ to itself that maps the standard coordinate “T” on $\mathbb{A}^1$ [regarded, say, as a rational function on $\mathbb{A}^1$] to T^{-1}; that is to say, such a “gluing without labels” results in a **contradiction** [i.e., since $T \neq T^{-1}$!], unless one passes to some sort of **quotient** of $\mathbb{A}^1$ — which amounts, from a foundational/logical point of view, to the introduction of some sort of **indeterminacy**, i.e., to the consideration of some sort of **collection of possibilities** [cf. “$\vee$”!].

(iii) The discussion of the projective line in (ii) is *truly remarkable* in that it **completely parallels** — i.e., relative to the *correspondence*

$$\text{“}-1\text{”} \qquad \longleftrightarrow \qquad \text{“}j^2\text{”}$$

between the *exponent “–1”* in the discussion of (ii) and the *exponents* “j^2”, where j ranges from 1 to $l^{\divideontimes}$, in the discussion of §3.4 — numerous aspects of the **Θ-link** of inter-universal Teichmüller theory, which we shall discuss in more detail in §3 [cf., especially, §3.4]. Indeed,

(iii-a) the isomorphism of (ii-a) may be understood as corresponding to the fact that the **($\Theta^{\pm\text{ell}}$NF-)Hodge theaters** on either side of the **Θ-link** in inter-universal Teichmüller theory are **isomorphic**, while

(iii-b) the observation of (ii-b) may be understood as corresponding to the fact that there is **no isomorphism** of **($\Theta^{\pm\text{ell}}$NF-)Hodge theaters** as in (iii-a) that maps the **Θ-pilot** in the *domain* of the Θ-*link* [which corresponds to “${}^{\dagger}q^{-1}$”] to the **$\boldsymbol{q}$-pilot** in the *codomain* of the Θ-*link* [which corresponds to “${}^{\ddagger}q$”].

On the other hand,

(iii-c) the isomorphism of (ii-c) may be understood as corresponding to the **full poly-isomorphism** of [**multiplicative!**] **$\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips** that constitutes the **Θ-link**, while

(iii-d) the observation of (ii-d) may be understood as corresponding to the fact that the full poly-isomorphism of (iii-c) is **not compatible** with the **ring structures** determined by the *($\Theta^{\pm\text{ell}}$NF-)Hodge theaters* on either side of the Θ-*link*, i.e., in particular, does **not** arise from a poly-isomorphism between these ($\Theta^{\pm\text{ell}}$NF-)Hodge theaters on either side of the Θ-link [cf. (iii-b)].

Next, we observe that

(iii-e) the gluing of (ii-e) may be understood as corresponding to the **gluing** constituted by the **Θ-link** between the ($\Theta^{\pm\text{ell}}$NF-)Hodge theaters on either side of the Θ-link, i.e., a gluing along [**multiplicative!**] **$\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips** that is **not compatible** with the **ring structures** in the *domain* and *codomain* of the Θ-link, but which allows one to obtain a **single $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip**, up to isomorphism, that may be interpreted

simultaneously as the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip arising from the **Θ-pilot** in the *domain* of the Θ-link **AND** [cf. "$\wedge$"!] as the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip arising from the ***q*-pilot** in the *codomain* of the Θ-link.

Here, we recall that this crucial **logical AND "$\wedge$" property** of the Θ-link is the *central theme* of the present paper [cf. the discussion of Examples 2.4.1, 2.4.2, 2.4.3, 2.4.4, 2.4.5, 2.4.6!]. Next, we observe that

(iii-f) the observation of (ii-f) may be understood as corresponding to the fact that, at least from an *a priori* point of view, there is **no natural way** to **express** the **Θ-pilot** of the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater in the *domain* of the Θ-link, relative to the gluing of (iii-e), in terms of the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater in the *codomain* of the Θ-link, while

(iii-g) the observation of (ii-g) may be understood as corresponding to the **simultaneous holomorphic expressibility (SHE)** property of the **multiradial representation** of the **Θ-pilot** [cf. [IUTchIII], Remark 3.11.1, (iii); [IUTchIII], Remark 3.9.5, (viii), (ix)], which allows one to **express** the **Θ-pilot** of the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater in the *domain* of the Θ-link, relative to the gluing of (iii-e), in terms of the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater in the *codomain* of the Θ-link [cf. also the discussion of (iv), (v), below; the discussion surrounding Example 2.4.8, (iii-a), (iii-b), below].

Finally, we note that

(iii-h) the *"gluing without labels"* discussed in (ii-h) may be understood as corresponding to the oversimplified version **"RCS-IUT"** of inter-universal Teichmüller theory obtained by implementing the RCS-identifications of RCS-redundant copies of objects [cf. the discussion of §1.2, Example 2.4.5], which leads to an immediate **contradiction**, unless one introduces some sort of **quotient/indeterminacy**, i.e., which amounts to the consideration of some sort of **collection of possibilities** [cf. "$\vee$"!].

In particular, relative to this **remarkably close structural resemblance** between the **gluing** that appears in the standard construction of the **projective line** and the **gluing** constituted by the **Θ-link** of inter-universal Teichmüller theory, the *central assertion*

$$\text{"IUT} \Leftrightarrow \text{RCS-IUT"}$$

of the RCS [cf. the discussion of Example 2.4.5] may be understood as corresponding to the assertion of an **"obvious equivalence"** [cf. the discussion of §1.3] between

· the **projective line**, on the one hand, and
· the **affine line** regarded up to some sort of identification of the standard coordinate "T" on the affine line with its inverse, on the other.

(iv) From the point of view of the analogy discussed in (iii) between the *gluing construction* of the *projective line* and *inter-universal Teichmüller theory*, perhaps the *closest nontrivial analogue*, in the case of the projective line, to the **multiradial representation** of the **Θ-pilot** in inter-universal Teichmüller theory is the group of **projective general linear** [i.e., "PGL_2"] **symmetries** of the projective line [cf. also the discussion of (v) below; the discussion of Example 3.10.1, (i), below].

That is to say, although there is also an analogy, discussed in (iii-g), with the observation of (ii-g), the content of (ii-g) is rather *formal/trivial*. By contrast, the PGL_2-symmetries of the projective line are *somewhat less trivial*, especially from the point of view of the *gluing construction* of the projective line discussed in (ii) [cf. also the discussion of (v) below; the discussion of Example 3.10.1, (i), below].

(v) The PGL_2-symmetries of the projective line are, in some sense, especially interesting in the case where one takes k to be the field $\mathbb{C}$ of *complex numbers*, and one restricts to the *subgroup*

$$PU_2 \quad \subseteq \quad PGL_2(\mathbb{C})$$

given by the image of the unitary matrices, i.e., the *projective unitary group*. Thus, as is well-known, one may think of PU_2 as the group of *isometric symmetries* of the Riemann surface associated to the projective line over $\mathbb{C}$ equipped with the *Fubini-Study metric*. The underlying topological space of this Riemann surface may be naturally identified with the *sphere* $\mathbb{S}^2$. The *geodesics* associated to the *Fubini-Study metric* then correspond to *great circles* on the sphere $\mathbb{S}^2$ [cf., e.g., the illustration of [GeoSph]]. In particular, the geodesics that pass through the *north/south poles* of $\mathbb{S}^2$ may be thought of as *lines of longitude*. In the current *metrized* situation, it is natural to think of $\mathbb{S}^2$ as being obtained *not* via a gluing of the complement of the north pole to the complement of the south pole [i.e., as in (ii-e)], but rather as being obtained via a gluing

$$\mathbb{S}^2 \quad \supseteq \quad \mathbb{H}^+ \quad \supseteq \quad \mathbb{E} \overset{\mathrm{def}}{=} \mathbb{H}^+ \cap \mathbb{H}^- \quad \subseteq \quad \mathbb{H}^- \quad \subseteq \quad \mathbb{S}^2$$

of the *northern hemisphere* $\mathbb{H}^+ \subseteq \mathbb{S}^2$ of $\mathbb{S}^2$ to the *southern hemisphere* $\mathbb{H}^- \subseteq \mathbb{S}^2$ of $\mathbb{S}^2$ along the *equator* $\mathbb{E} \subseteq \mathbb{S}^2$. Here, we note that the *gluing* of (ii-e) — which yields a *single rational function* on the projective line that corresponds *simultaneously* to the standard coordinate "${}^\dagger T$" on ${}^\dagger\mathbb{A}^1$ **AND** to the standard coordinate "${}^\ddagger T^{-1}$" on ${}^\ddagger\mathbb{A}^1$ — may be thought of, in the current *metrized* situation, as corresponding to the following [at first glance, *self-contradictory*!] phenomenon:

(OrFlw) an **oriented flow** along the **equator** — which may be thought of physically as a sort of *wind current* — that flows from *east* to *west* appears *simultaneously* to be flowing, from the point of view of the *northern hemisphere* $\mathbb{H}^+ \subseteq \mathbb{S}^2$, in the *clockwise* direction **AND**, from the point of view of the *southern hemisphere* $\mathbb{H}^- \subseteq \mathbb{S}^2$, in the *counterclockwise* direction.

Next, let us recall that — unlike $\mathbb{S}^2$! — both $\mathbb{H}^+$ and $\mathbb{H}^-$ may be thought of as *closed discs* in the plane. Thus, in summary,

(GdsFlw) the geodesic geometry of the **Fubini-Study metric** — i.e., in essence, the $(PGL_2(\mathbb{C}) \supseteq)$ PU_2**-symmetries** of $\mathbb{S}^2$ — allow one, by considering the **geodesic flow** along **lines of longitude**, to **represent**, up to a *relatively mild distortion*, the entirety of $\mathbb{S}^2$, i.e., *including* $\mathbb{H}^- \subseteq \mathbb{S}^2$, as a sort of **extension/deformation** of the closed disc $\mathbb{H}^+$.

Indeed, (GdsFlw) is precisely the principle that is applied to represent, using *lines of longitude*, the **globe** [i.e., in the sense of the surface of the planet earth] via a rectangular, planar, cartesian **map** [i.e., in the sense of cartography]! Note, moreover, that

(NoLbDlt) although the approach of (GdsFlw) gives rise to a certain relatively mild degree of *distortion* in the **representation** of $\mathbb{H}^-$ in terms of $\mathbb{H}^+$, it does **not** involve any sort of **naive identification** of the closed discs $\mathbb{H}^+$, $\mathbb{H}^-$, i.e., any sort of **arbitrary label deletion**, in the style of (ii-h).

The interpretation discussed in (GdsFlw) and (NoLbDlt) of the ($PGL_2(\mathbb{C}) \supseteq$) PU_2-**symmetries** of $\mathbb{S}^2$ may be understood as strongly suggesting a *nontrivial analogy* between these symmetries of $\mathbb{S}^2$ and the **multiradial representation** of the Θ-**pilot** in inter-universal Teichmüller theory [cf. the analogy between *multiradiality* and *connections/parallel transport/crystals* discussed in [Alien], §3.1, (iv), (v), as well as §3.5, §3.10, below].

Example 2.4.8: Gluings of rings along multiplicative monoids.

(i) Let R be an *integral domain* equipped with the action of a *group* G and N a *positive integer* ≥ 2. For simplicity, we assume that $N = 1 + \cdots + 1$ [i.e., the sum of N copies of "$1 \in R$"] determines a *nonzero element* of R. Write

- $R^{\rhd} \subseteq R$ for the *multiplicative monoid* of nonzero elements of R;
- $R^{\rhd} \twoheadrightarrow R^{\rhd\mu}$ for the *quotient multiplicative monoid* of $R^{\rhd}$ by the group of *roots of unity* of R;
- $(R^{\rhd})^N \subseteq R^{\rhd}$, $(R^{\rhd\mu})^N \subseteq R^{\rhd\mu}$ for the *multiplicative submonoids* consisting of the N*-th powers* of elements of "$(-)$".

Thus, G acts naturally and in a compatible fashion not only on the *ring* R, but also on the *multiplicative monoids* $R^{\rhd}$, $R^{\rhd\mu}$, $(R^{\rhd})^N$, $(R^{\rhd\mu})^N$. Also, we observe that the N*-th power map* on $R^{\rhd\mu}$ determines an *isomorphism of multiplicative monoids* equipped with actions by G

$$R^{\rhd\mu} \quad \overset{\sim}{\rightarrow} \quad (R^{\rhd\mu})^N \ (\subseteq R^{\rhd\mu})$$

that does **not arise** from a **ring homomorphism**, i.e., as may be seen from the fact that this isomorphism of multiplicative monoids is *not compatible* with the operation of *addition* [cf. our assumption that N determines a *nonzero element* of R!] .

(ii) Let ${}^{\dagger}R$, ${}^{\ddagger}R$ be *two distinct copies* of the integral domain R of (i), equipped with respective actions by *two distinct copies* ${}^{\dagger}G$, ${}^{\ddagger}G$ of the group G of (i). We shall use similar notation for objects labeled with "†" or "‡" to the notation introduced in (i) for objects not equipped with such labels. Then

(ii-a) one may use the *isomorphism of multiplicative monoids* arising from the N*-th power map* discussed in (i) to **glue** together

$$ {}^{\dagger}G \curvearrowright {}^{\dagger}R \supseteq ({}^{\dagger}R^{\rhd})^N \twoheadrightarrow ({}^{\dagger}R^{\rhd\mu})^N \quad \overset{\sim}{\leftarrow} \quad {}^{\ddagger}R^{\rhd\mu} \twoheadleftarrow {}^{\ddagger}R^{\rhd} \subseteq {}^{\ddagger}R \curvearrowleft {}^{\ddagger}G $$

the **ring** ${}^{\dagger}R$ *to the* **ring** ${}^{\ddagger}R$ *along the* **multiplicative monoid** $({}^{\dagger}R^{\rhd\mu})^N \overset{\sim}{\leftarrow} {}^{\ddagger}R^{\rhd\mu}$ [cf. the discussion of Example 2.4.7, (ii), (iii)!].

Here, we observe that this gluing is *compatible* with the respective actions of ${}^{\dagger}G$, ${}^{\ddagger}G$ relative to the isomorphism ${}^{\dagger}G \overset{\sim}{\rightarrow} {}^{\ddagger}G$ given by forgetting the labels "†", "‡", but in this context, it is of the *utmost importance* to remember that (ii-b) since, as observed in (i), the N-th power map is *not compatible* with the operation of *addition* (!), this isomorphism ${}^{\dagger}G \xrightarrow{\sim} {}^{\ddagger}G$ may be regarded either as an *isomorphism of* **abstract groups** or as an *isomorphism of groups equipped with actions on certain* **multiplicative monoids**, but **not** as an isomorphism of groups equipped with actions on *rings* [i.e., ${}^{\dagger}R$, ${}^{\ddagger}R$],

e.g., as is the case where ${}^{\dagger}G$, ${}^{\ddagger}G$ are taken to be **"Galois groups"** [that is to say, groups equipped with faithful actions on some *field*, such as the quotient field of ${}^{\dagger}R$ or ${}^{\ddagger}R$]. In the context of (ii-b), we observe that, of course, one may also consider taking the point of view that ${}^{\dagger}G$, ${}^{\ddagger}G$ are *groups equipped with actions on the* **diagram**

$${}^{\dagger}R \supseteq ({}^{\dagger}R^{\rhd})^N \twoheadrightarrow ({}^{\dagger}R^{\rhd\mu})^N \quad \xleftarrow{\sim} \quad {}^{\ddagger}R^{\rhd\mu} \twoheadleftarrow {}^{\ddagger}R^{\rhd} \subseteq {}^{\ddagger}R$$

[consisting of various rings, multiplicative monoids, etc.] of (ii-a), i.e., *not* just on some *isolated portion* of the diagram such as ${}^{\dagger}R$, $({}^{\dagger}R^{\rhd\mu})^N$, ${}^{\ddagger}R^{\rhd\mu}$, or ${}^{\ddagger}R$.

(ii-c) The *fundamental* — and indeed *essentially tautological*! — problem, however, with this approach of thinking of ${}^{\dagger}G$, ${}^{\ddagger}G$ as *groups of automorphisms* of the diagram of the above display is that this approach yields a situation in which one can *no longer consider* [i.e., in the sense that it is *no longer a well-defined proposition* to consider!] various *isolated portions* of the diagram [i.e., such as ${}^{\dagger}R$, $({}^{\dagger}R^{\rhd})^N$, ${}^{\ddagger}R^{\rhd\mu}$, or ${}^{\ddagger}R$] equipped with actions by ${}^{\dagger}G$, ${}^{\ddagger}G$ *independently* of the entire diagram.

On the other hand, as we shall see in (iii) below, the *main issue of interest* surrounding the gluing of (ii-a) involves consideration of the extent to which one can start precisely from such an *isolated portion* of the diagram — namely, the *glued data*

$${}^{\dagger}G \curvearrowright ({}^{\dagger}R^{\rhd\mu})^N \xleftarrow{\sim} {}^{\ddagger}R^{\rhd\mu} \curvearrowleft {}^{\ddagger}G$$

— and then proceed to reconstruct, possibly *up to relatively mild indeterminacies*, some *remaining portion* of the diagram. Finally, we observe that the importance, in the context of inter-universal Teichmüller theory, of thinking of Galois groups [*not* as groups of automorphisms of ring/fields/diagrams involving rings (!), but rather] as **abstract groups** [i.e., as emphasized in the above discussion!] is reminiscent of the discussion of the issue of the *"relative subordination"* of *group theory* versus *field theory* [i.e., "group theory $\ggg$ field theory" versus "field theory $\ggg$ group theory"] in [Alien], §4.4, (i).

(iii) In general, in the situation of the **gluing** considered in (ii-a),

(iii-a) the problem of *describing the* **additive structure** *of* ${}^{\dagger}R$ *in terms of the additive structure of* ${}^{\ddagger}R$ — in a fashion that is compatible with the **gluing** and via a **single** algorithm that may be applied to the *glued data* to reconstruct **simultaneously** the additive structures of **both** ${}^{\dagger}R$ and ${}^{\ddagger}R$ — seems to be *hopelessly intractable*!

The nontriviality of this problem may already be seen, for instance, in the case where one takes R to be $\mathbb{Z}$ [i.e., the ring of rational integers]. Indeed, this sort of problem may be understood as (iii-b) the *starting point* of inter-universal Teichmüller theory, where one considers the **gluing** constituted by the **Θ-link** [cf. the discussion of §3.4 below] and the issue of describing — in a fashion compatible with the **crucial logical AND** property [cf. the discussion of (iv) below] associated to this *gluing*! — certain portions of the **ring/additive structure** of the *domain* [i.e., labeled by "†"] of the Θ-link in terms of the **ring/additive structure** of the *codomain* [i.e., labeled by "‡"] of the Θ-link via a **single** algorithm that may be applied to the *glued data* to reconstruct **simultaneously** the corresponding portions of the ring/additive structure of **both** the *domain* and the *codomain* of the Θ-link [cf. the discussion of the **simultaneous holomorphic expressibility (SHE)** property in [IUTchIII], Remark 3.11.1, (iii); [Alien], §3.7, (i); [Alien], §3.11, (iv)].

Such a description is ultimately achieved in inter-universal Teichmüller theory by means of the **multiradial representation** of the **Θ-pilot**, which allows one to reconstruct, up to *relatively mild indeterminacies*, certain portions of interest of the ring/additive structure of the *domain* of the Θ-link in terms of the ring/additive structure of the *codomain* of the Θ-link [cf. the discussion of Example 2.4.7, (v)] — in a fashion that is compatible with the *gluing* and via a *single* algorithm that may be applied to the *glued data* to reconstruct *simultaneously* the corresponding portions of the ring/additive structure of *both* the *domain* and the *codomain* of the Θ-link — by making use of certain structural properties of the various *multiplicative monoids equipped with group actions* that appear in the construction of the Θ-link, as well as certain highly nontrivial **anabelian** properties of the underlying **abstract groups** of the various Galois groups that appear [cf. the discussion of (ii-b) above; the discussion of §3.2, §3.8, below]. In this context, it is also interesting to note that, when $N = p$ is a *prime number*, the fact that the **Frobenius morphism** given by raising to the power p is a *ring homomorphism* in characteristic p may be interpreted in the following way:

(iii-c) even in the situation of the present discussion [i.e., where the ring R is *not* of positive characteristic!], the *isomorphism of multiplicative monoids* obtained by raising to the p-th power — i.e., the isomorphism of multiplicative monoids that appears in the *gluing* of (ii-a) — may in fact be regarded as being **"simultaneously compatible"** with the **additive structures** in its domain and codomain if one regards one's computations as being subject to the **"indeterminacy"** given by working *modulo* p.

Finally, we observe that this interpretation is reminiscent of the important analogies between inter-universal Teichmüller theory, on the one hand, and *Frobenius liftings* and *p-adic Teichmüller theory*, on the other, as discussed in [Alien], §2.4, §2.5; [Alien], §3.3, (ii) [cf. also the discussion of **crystals** in [Alien], §3.1, (v), as well as §3.5, §3.10 below].

(iv) In the context of the gluing of (ii-a), we observe that

(iv-a) the *glued multiplicative monoid* $({}^{\dagger}R^{\rhd\mu})^N \xleftarrow{\sim} {}^{\ddagger}R^{\rhd\mu}$, regarded up to isomorphism, is **simultaneously**

- the *multiplicative monoid* $({}^{\dagger}R^{\rhd\mu})^N$ associated to the *ring* ${}^{\dagger}R$

AND

- the *multiplicative monoid* ${}^{\ddagger}R^{\rhd\mu}$ associated to the *ring* ${}^{\ddagger}R$.

In a similar vein, if one thinks of the *glued group* ${}^{\dagger}G \overset{\sim}{\to} {}^{\ddagger}G$, regarded up to isomorphism, either as an *abstract group* or as a *group equipped with an action on the glued multiplicative monoid*, then

(iv-b) this glued group ${}^{\dagger}G \overset{\sim}{\to} {}^{\ddagger}G$ is **simultaneously**

- the group ${}^{\dagger}G$ equipped with an action on the *multiplicative monoid* $({}^{\dagger}R^{\rhd\mu})^N$ associated to the *ring* ${}^{\dagger}R$

AND

- the group ${}^{\ddagger}G$ equipped with an action on the *multiplicative monoid* $({}^{\dagger}R^{\rhd\mu})^N$ associated to the *ring* ${}^{\ddagger}R$.

The gluing given by the **Θ-link** involves entirely analogous **logical AND** properties, which are *fundamental* to the *essential logical structure of inter-universal Teichmüller theory*, as exposed in the present paper. Of course,

(iv-c) it is always possible to consider the situation in which one **deletes** the **labels** "†", "‡", but only at the expense of **sacrificing** these **crucial logical AND** properties (iv-a), (iv-b), i.e., at the expense of agreeing to work under the assumption that the "glued data" is the data associated to "†" **OR** the data associated to "‡", but *not necessarily both simultaneously.*

Moreover, once one *deletes* the *labels* "†", "‡" — i.e., so that the two copies of "R" are **identified** with one another via an *isomorphism of rings*! — the problem of describing the ring/additive structure of one copy in terms of the ring/additive structure of the other copy [cf. the discussion of (iii)!] becomes **"trivial"**, but this triviality is of little interest since it is achieved only at the cost of sacrificing the **crucial logical AND** properties in favor of the [entirely uninteresting!] **logical OR** property just described — cf. the discussion surrounding (NeuORInd) in §3.4 below, as well as the discussion of

IUT = AND-IUT

versus **RCS-IUT/OR-IUT/EssOR-IUT**

in Examples 2.4.5, (ii), (iii), (v), (vi), (vii); 2.4.7, (ii), (iii), (v).

(v) Finally, we note that the relationship between the discussion of the present Example 2.4.8 and the *numerical situation* discussed in Example 2.4.5, (ii), (iii) [cf. also the discussion of the *final page and a half* of the files "[SS2018-05]", "[SS2018-08]" available at the website [Dsc2018]] may be seen by considering the case where R is taken to be the *ring of rational integers* $\mathbb{Z}$ [or, in fact, slightly more generally, the *ring of integers* "$\mathcal{O}_F$" of a finite field extension F of the *field of rational numbers* $\mathbb{Q}$ — a situation that may be related to the case of $\mathbb{Z}$ by applying the multiplicative *norm map* $\mathbb{N}_{F/\mathbb{Q}} : F \to \mathbb{Q}$]. Indeed, in the case where R is taken to be $\mathbb{Z}$, one may consider the **"height"**

$$\log(|x|) \in \mathbb{R}$$

[where "log" denotes the *natural logarithm* of a positive real number, and "$| - |$" denotes the absolute value of an element of $\mathbb{Z}$] associated to a nonzero element $0 \neq x \in \mathbb{Z}$. Then the *N-th power map* of (i), (ii) corresponds, after passing to *heights*, to *multiplying real numbers by N*, i.e., in essence to the situation considered in Example 2.4.5, (ii), (iii) [which corresponds to the case where the "N" of the present discussion is taken to be 2].

Section 3: The logical structure of inter-universal Teichmüller theory

In the present §3, we give a detailed exposition of the **essential logical structure** *of inter-universal Teichmüller theory*, with a special focus on issues related to *RCS-redundancy.* From a strictly rigorous point of view, this exposition assumes a substantial level of knowledge and understanding of the technical details of inter-universal Teichmüller theory [which are surveyed, for instance, in [Alien]]. On the other hand, in a certain *qualitative sense*, the discussion of the present §3 may in fact be understood, at a relatively elementary level, via the analogies that we discuss with the topics covered in §2. Indeed, in this context, it should be emphasized that, despite the relatively novel nature of the set-up of inter-universal Teichmüller theory,

> the *essential mathematical content* that lies at the heart of all of the issues covered in the present §3 concerns entirely well-known mathematics at the *advanced undergraduate or beginning graduate* level [i.e., the topics covered in §2].

§3.1. One-dimensionality via identification of RCS-redundant copies

Inter-universal Teichmüller theory concerns the explicit description of the relationship between various possible *intertwinings* — namely,

the **"Θ"-** and **"$\boldsymbol{q}$-" intertwinings**

— between the **two underlying combinatorial/arithmetic dimensions** of a ring [cf., e.g., [Alien], §2.11; [Alien], §3.11, (v), as well as the discussion of §3.9 below]. There are many different ways of thinking about these two underlying combinatorial/arithmetic dimensions of a ring; one way to understand these two dimensions is to think of them as corresponding, respectively, to the **unit group** and **value group** of the various local fields that appear as completions of a number field at one of its valuations.

In more technical language, this sort of decomposition into unit groups and value groups may be seen in the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strips** that appear in the **Θ-link** of inter-universal Teichmüller theory. Thus, if one thinks in terms of such $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips, then inter-universal Teichmüller theory may be summarized as follows:

(2-Dim) The main content of inter-universal Teichmüller theory consists of an **explicit description**, up to certain relatively mild indeterminacies, of the **Θ-intertwining** on the [*two-dimensional!*] $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strips** that appear in the **Θ-link** in terms of the **$\boldsymbol{q}$-intertwining** on these $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips by means of the **$\mathfrak{log}$-link** and various types of **Kummer**

theory that are used to relate **Frobenius-like** and **étale-like** structures [cf. the discussion of Example 2.4.8, (iii)].

In particular, the essential mathematical content of inter-universal Teichmüller theory concerns an *a priori* **variable** relationship between the **two underlying combinatorial/arithmetic dimensions** of a ring.

Put another way, if one arbitrarily *"crushes"* these two dimensions into a *single dimension* — i.e., in more technical language, assumes that

(1-Dim) there exists a **consistent choice** of a **fixed relationship** between these two dimensions of (2-Dim), so that these two dimensions may, in effect, be regarded as a **single dimension**

— then one immediately obtains a *superficial contradiction* [cf. the discussion of Example 3.1.1, (i-b), (ii-b), below]. Indeed, this is one of the *central assertions* of the RCS [cf. the discussion following Example 3.1.1]. This is not a "new" observation, but rather, in some sense, the *starting point* of inter-universal Teichmüller theory, i.e., the *initial motivation* for regarding the relationship between the two underlying combinatorial/arithmetic dimensions of a ring as being *variable*, rather than fixed.

__Example 3.1.1__**: Elementary models of gluings and intertwinings.** In the following, we shall write V for the topological group $\mathbb{R}_{>0}$. Let $x, y \in V$ be [not necessarily distinct!] elements of V and $Y \subseteq V$ a nonempty subset of V. Let V^{rl}, V^{im} be *two distinct labeled copies* of V, which we think of as corresponding to the positive portions of the *real* and *imaginary* axes in the complex plane.

(i) Let ${}^{\dagger}V$, ${}^{\ddagger}V$ be two **not necessarily distinct copies** of V. We shall write ${}^{\dagger}y \in {}^{\dagger}V$, ${}^{\ddagger}x \in {}^{\ddagger}V$ for the respective elements determined by $y, x \in V$.

(i-a) Suppose that ${}^{\dagger}V$, ${}^{\ddagger}V$ are **distinct copies** of V. Write W for the topological space obtained by **gluing** ${}^{\dagger}V$, ${}^{\ddagger}V$ along the homeomorphic subspaces $\{{}^{\dagger}y\} \subseteq {}^{\dagger}V$, $\{{}^{\ddagger}x\} \subseteq {}^{\ddagger}V$. Then observe that this construction of W is *well-defined* and *free* of any *internal contradictions.* Moreover, the existence of W does **not imply** any nontrivial conclusions concerning x and y.

Note the *sharp contrast* between the situation discussed in (i-a) and the following situation:

(i-b) Suppose that ${}^{\dagger}V$, ${}^{\ddagger}V$ are in fact the **same copy** of V, i.e., ${}^{*}V \overset{\mathrm{def}}{=} {}^{\dagger}V = {}^{\ddagger}V$. Consider the *assertion* that

> the topological space ${}^{*}V$ is obtained by **gluing** ${}^{\dagger}V$, ${}^{\ddagger}V$ along the homeomorphic subspaces $\{{}^{\dagger}y\} \subseteq {}^{\dagger}V$, $\{{}^{\ddagger}x\} \subseteq {}^{\ddagger}V$.

Then observe that this *assertion* concerning ${}^{*}V$ is *well-defined* and *free* of *internal contradictions* **only** in the case where $x = y$. That is to say, the existence of a topological space ${}^{*}V$ as described in the above *assertion* **implies** the *nontrivial conclusion* that $x = y$, or, equivalently, a **"contradiction"** to the assertion that $x \neq y$.

One may also consider the following *variant* of (i-b):

(i-c) One replaces $\{{}^{\dagger}y\} \subseteq {}^{\dagger}V$ in (i-b) by the nonempty subset ${}^{\dagger}Y \subseteq {}^{\dagger}V$ [i.e., determined by $Y \subseteq V$], where one thinks of this subset as a set of *"possible *y's"*. The resulting *"assertion"* then becomes a corresponding collection of *assertions* related by *logical OR "$\vee$'s"*, and the final **nontrivial conclusion** is that $x \in Y$.

(ii) The elementary models presented in (i) may be interpreted as essentially equivalent representations of various models of **"holomorphic structures"** [cf. the discussion below of (InfH), as well as Examples 3.3.1, 3.3.2] — i.e., in the terminology of the discussion preceding the present Example 3.1.1, **"intertwinings"** — between the *"real"* and *"imaginary" dimensions* V^{rl}, V^{im}. Here, we think of "holomorphic structures"/"intertwinings" as being defined by assignments

$$V^{\mathrm{rl}} \ni 1^{\mathrm{rl}} \quad \mapsto \quad ? \in V^{\mathrm{im}}$$

[where $1^{\mathrm{rl}} \in V^{\mathrm{rl}}$ denotes the element determined by $1 \in V$], corresponding to *"counterclockwise rotations by* 90 *degrees"*, or, alternatively, *"multiplication by* $\sqrt{-1}$*"*. Indeed, let $^{\dagger}V^{\mathrm{rl}}$, $^{\ddagger}V^{\mathrm{rl}}$ be two **not necessarily distinct copies** of V^{rl}; $^{\dagger}V^{\mathrm{im}}$, $^{\ddagger}V^{\mathrm{im}}$ two **not necessarily distinct copies** of V^{im}. We shall write $^{\dagger}y^{\mathrm{im}} \in {}^{\dagger}V^{\mathrm{im}}$, $^{\ddagger}x^{\mathrm{im}} \in {}^{\ddagger}V^{\mathrm{im}}$ for the respective elements determined by $y, x \in V$. Then the discussion of (i-a) may be translated into a discussion concerning *intertwinings* by arguing as follows:

(ii-a) Suppose that $^{\dagger}V^{\mathrm{rl}}$, $^{\ddagger}V^{\mathrm{rl}}$ are **distinct copies** of V^{rl}; $^{\dagger}V^{\mathrm{im}}$, $^{\ddagger}V^{\mathrm{im}}$ are **distinct copies** of V^{im}. Here, we think of $^{\dagger}V^{\mathrm{rl}}$, $^{\dagger}V^{\mathrm{im}}$ as being equipped with the *intertwining* given by taking "?" to be $^{\dagger}y^{\mathrm{im}} \in {}^{\dagger}V^{\mathrm{im}}$; we think of $^{\ddagger}V^{\mathrm{rl}}$, $^{\ddagger}V^{\mathrm{im}}$ as being equipped with the *intertwining* given by taking "?" to be $^{\ddagger}x^{\mathrm{im}} \in {}^{\ddagger}V^{\mathrm{im}}$. Then one applies (i-a), relative to the correspondences $^{\dagger}V^{\mathrm{im}} \longleftrightarrow {}^{\dagger}V$, $^{\ddagger}V^{\mathrm{im}} \longleftrightarrow {}^{\ddagger}V$. This yields a **gluing** as in (i-a) that is *well-defined* and *free* of any *internal contradictions*. Moreover, the existence of such a gluing does **not imply** any nontrivial conclusions concerning x and y.

In a similar vein:

(ii-b) Suppose that $^{\dagger}V^{\mathrm{rl}}$, $^{\ddagger}V^{\mathrm{rl}}$ are in fact the **same copy** of V^{rl}, i.e., $^{*}V^{\mathrm{rl}} \stackrel{\mathrm{def}}{=} {}^{\dagger}V^{\mathrm{rl}} = {}^{\ddagger}V^{\mathrm{rl}}$, and that $^{\dagger}V^{\mathrm{im}}$, $^{\ddagger}V^{\mathrm{im}}$ are in fact the **same copy** of V^{im}, i.e., $^{*}V^{\mathrm{im}} \stackrel{\mathrm{def}}{=} {}^{\dagger}V^{\mathrm{im}} = {}^{\ddagger}V^{\mathrm{im}}$. Then one applies (i-b), relative to the correspondence $^{*}V^{\mathrm{im}} \longleftrightarrow {}^{*}V$. This yields an *assertion* concerning a **gluing** as in (i-b) — i.e., in the language of the present discussion, concerning a *coincidence* of the *intertwining* on $^{\dagger}V^{\mathrm{rl}}$, $^{\dagger}V^{\mathrm{im}}$ with the *intertwining* on $^{\ddagger}V^{\mathrm{rl}}$, $^{\ddagger}V^{\mathrm{im}}$ — that is *well-defined* and *free* of *internal contradictions* **only** in the case where $x = y$. That is to say, the existence of such a gluing **implies** the *nontrivial conclusion* that $x = y$, or, equivalently, a **"contradiction"** to the assertion that $x \neq y$.

(ii-c) One replaces $\{^{\dagger}y^{\mathrm{im}}\} \subseteq {}^{\dagger}V^{\mathrm{im}}$ in (ii-b) [cf. also the notation of (ii-a)] by the nonempty subset $^{\dagger}Y^{\mathrm{im}} \subseteq {}^{\dagger}V^{\mathrm{im}}$ [i.e., the subset determined by $Y \subseteq V$], where one thinks of this subset as a set of *"possible y's"*. The resulting *"assertion"* then becomes a corresponding collection of *assertions* related by *logical OR "$\vee$'s"*, and the final **nontrivial conclusion** is that $x \in Y$.

(iii) Relative to the analogy with *inter-universal Teichmüller theory*, we have correspondences with objects that appear in the elementary models of (ii) as follows:

$$V^{\mathrm{rl}} \longleftrightarrow \text{ the } \textit{value group} \text{ portion of an } \mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}\text{-prime-strip;}$$

$$V^{\mathrm{im}} \longleftrightarrow \text{ the } \textit{unit group} \text{ portion of an } \mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}\text{-prime-strip;}$$

$$\dagger/\ddagger \longleftrightarrow (\Theta^{\pm\mathrm{ell}}\text{NF-)Hodge theaters in the } \textit{domain/codomain} \text{ of the } \Theta\text{-link;}$$

$$\text{intertwinings involving ``}y\text{''} \longleftrightarrow \text{ the } \Theta\textit{-intertwinings;}$$

$$\text{intertwinings involving ``}x\text{''} \longleftrightarrow \text{ the } q\textit{-intertwinings;}$$

[cf. the discussion at the beginning of §3.4]. Here, we note that from the point of view of *intertwinings*, the *unit group* portion corresponding to "V^{im}" must be understood as being **log-shifted** by -1, relative to the *value group* portion corresponding to "V^{rl}" [cf. the discussion below of (InfH), as well as Examples 3.3.1, 3.3.2]. That is to say, if the *value group* portion corresponding to "V^{rl}" is located at the coordinate (n, m) of the *log-theta-lattice*, then the *unit group* portion corresponding to "V^{im}" must be understood as being located at the coordinate $(n, m-1)$ of the log-theta-lattice. In particular, the *unit group* and *value group* portions corresponding to a *pair* "$(V^{\mathrm{rl}}, V^{\mathrm{im}})$" belong to *different* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips*. From the point of view of the discussion of (1-Dim), the "*consistent choice* of a *fixed relationship*" corresponds to the **coincidence** of **intertwinings** in (ii-b), while the resulting *"superficial contradiction"* corresponds to the **"contradiction"** discussed in (ii-b). On the other hand, the *"explicit description"/"variable relationship"* of (2-Dim), which leads naturally to a *numerical estimate/inequality* concerning *log-volumes* [cf. Example 2.4.5, (ii)], corresponds to the situation involving *various possibilities* discussed in (ii-c), which leads to the **nontrivial conclusion** "$x \in Y$" [cf. the discussion of **"closed loops"** in (Stp7), (Stp8) of §3.10 below; the discussion of (DltLb) in §3.11 below; the discussion of [IUTchIII], Remark 3.12.2, (ii)].

(iv) Finally, we observe in passing that the **fixed intertwining** of (ii-b) [cf. also the discussion of (ii-b) in (iii), as well as the discussion of (FxRng), (FxEuc), (FxFld), (RdVar) below] may be regarded as being analogous to the well-known classical **holomorphic** approach to the theory of **moduli** of [one-dimensional] **complex tori**, that is to say, in which one works with a copy of the *upper half-plane* "$\mathfrak{H}$" with a *fixed holomorphic structure* and thinks of the moduli of complex tori as a *"variation of period matrices"* [i.e., the holomorphic parameter "$z \in \mathfrak{H}$", which may be taken, in the notation of (ii-b), to be "ix" or "iy"]. By contrast, the situation involving the set "${}^{\dagger}Y^{\mathrm{im}} \subseteq {}^{\dagger}V^{\mathrm{im}}$" discussed in (ii-c) may be regarded as analogous to the [real analytic] **Teichmüller** approach to the theory of moduli of complex tori [cf. the discussion of Example 3.3.1], i.e., in which the *holomorphic structure* is subject to *Teichmüller dilations* [corresponding to various elements in the set ${}^{\dagger}Y^{\mathrm{im}}$], relative to the *fixed "real analytic" pair* given by ${}^{\dagger}V^{\mathrm{rl}}$, ${}^{\dagger}V^{\mathrm{im}}$.

One *central assertion* of the RCS — which appears, for instance, in certain 10pp. manuscripts written by adherents of the RCS [cf., especially, the discussion of the *final page and a half* of the files "[SS2018-05]", "[SS2018-08]" available at the website [Dsc2018]] — is to the effect that the *existence*, as in (1-Dim), of a *consistent choice of a fixed relationship* between the two dimensions of (2-Dim) may be derived as a consequence — i.e., in more succinct notation,

$$\text{(RC-FrÉt), (RC-}\mathfrak{log}\text{), (RC-}\Theta\text{)} \quad \text{“}\Longrightarrow\text{”} \quad \text{(1-Dim)}$$

— of certain **“redundant copies assertions”**, as follows:

(RC-FrÉt) the **Frobenius-like** and **étale-like** versions of objects in inter-universal Teichmüller theory are **“redundant”**, i.e., may be **identified** with one another without affecting the essential logical structure of the theory;

(RC-$\mathfrak{log}$) the **($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters** on either side of the **$\mathfrak{log}$-link** in inter-universal Teichmüller theory are **“redundant”**, i.e., may be **identified** with one another without affecting the essential logical structure of the theory;

(RC-Θ) the **($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters** on either side of the **Θ-link** in inter-universal Teichmüller theory are **“redundant”**, i.e., may be **identified** with one another without affecting the essential logical structure of the theory.

In the remainder of the present §3 [cf., especially, §3.2, §3.3, §3.4], we discuss in more detail the **falsity** of each of these “RCS-redundancy” assertions [i.e., (RC-FrÉt), (RC-$\mathfrak{log}$), (RC-Θ)].

Here, it should be noted that this *falsity* of (RC-FrÉt), (RC-$\mathfrak{log}$), (RC-Θ) is *by no means a difficult or subtle issue*, but rather a sort of matter of *“belaboring the intuitively obvious”* from the point of view of mathematicians who are thoroughly familiar with inter-universal Teichmüller theory. Nevertheless, as discussed in [Rpt2018], §17, it is a *pedagogically meaningful exercise* to write out and discuss the details surrounding this sort of issue [cf. also the discussion in the final portions of §1.6, §1.7 of the present paper]. Moreover, as discussed in §1.5 of the present paper, it is desirable from a *historical* point of view to produce detailed, explicit, and readily accessible written expositions concerning this sort of issue.

This state of affairs prompts the following *question*:

> *Why do adherents of the RCS continue to insist on asserting the validity of these assertions (RC-FrÉt), (RC-$\mathfrak{log}$), (RC-Θ)?*

Any sort of complete, definitive answer to this question lies beyond the scope of the present paper. On the other hand, it seems natural to conjecture that one *fundamental motivation* for these assertions of RCS-redundancy may be found in the fact that

(FxRng) many arithmetic geometers have *only experienced* working in situations where **all schemes** — or, alternatively, **rings** — that appear in a theory are regarded as belonging to a **single category** that is **fixed** throughout the theory, hence are related to another via **ring homomorphisms**, i.e., in such a way that the **ring structure** of the various rings involved is always respected [cf. the discussion of §1.5, as well as the discussion of §3.8 below].

It is not difficult to imagine that the *heuristics and intuition* that result from years [or decades!] of immersive experience in such mathematical situations could create a *mindset* that is fertile ground for the RCS-redundancy assertions that will be discussed in detail in the remainder of the present §3 [cf., especially, §3.2, §3.3, §3.4].

Finally, we observe that this situation is, in certain respects, reminiscent of various situations that occurred throughout the **history of mathematics**, such as, for instance, the situation that occurred in the late 19-th century with regard to such novel [i.e., at the time] notions as the notion of an **abstract manifold** or an **abstract Riemann surface**. That is to say,

(FxEuc) from the point of view of anyone for whom it is a *"matter of course"* or *"common sense"* that all geometry must take place within *some* **fixed, static ambient Euclidean space** — such as, for instance, the **complex plane** — such *abstract geometric notions* as the notion of an abstract manifold or abstract Riemann surface might come across as *deeply disturbing* and *unlikely to be of use in any substantive mathematical sense* [cf. the discussion of §1.5; the discussion of [IUTchI], §I2].

In this context, it is of interest — especially from a historical point of view — to recall that, in some sense, the *most fundamental classical example* of such an abstract geometry is the Riemann surface that arises by applying the technique of **analytic continuation** to the **complex logarithm**, i.e., which may be regarded as a sort of *distant ancestor* [cf. the discussion of [IUTchI], Remark 5.1.4; [Alien], §3.3, (vi)] of the **$\mathfrak{log}$-link** of inter-universal Teichmüller theory. Another [in fact closely related!] *fundamental classical example* of such an abstract geometry is the **hyperbolic geometry** of the **upper half-plane**, which may also be regarded as a sort of *distant ancestor* of numerous aspects of inter-universal Teichmüller theory [cf. (InfH) and Example 3.3.2 in §3.3 below, as well as the discussion of [IUTchI], Remark 6.12.3, (iii); [IUTchIII], Remark 2.3.3, (ix), (x)].

Another *historically important* instance of this sort of situation may be seen in the introduction, in the early 19-th century, of **Galois groups** — i.e., of [finite] automorphism groups of **abstract fields** — as a tool for investigating the roots of polynomial equations. That is to say,

(FxFld) until the advent of *Galois groups/abstract fields*, the issue of investigating the roots of polynomial equations was always regarded — again as a *"matter of course"* or *"common sense"* — as an issue of investigating various *"exotic numbers"* inside *some* **fixed, static ambient field** such as the field of complex numbers; moreover, from this more classical "common sense" point of view, the idea of working with *automorphisms of abstract fields* — i.e., fields that are *not constrained* [since such constraints would rule out the existence of nontrivial automorphisms!] to be treated as subsets of some fixed, static ambient field — might come across as *deeply disturbing* and *unlikely to be of use in any substantive mathematical sense* [cf. the discussion of §1.5].

On the other hand, from the point of view of inter-universal Teichmüller theory, this *radical transition*

$$\textbf{roots as concrete numbers} \quad \rightsquigarrow \quad \textbf{Galois groups/abstract fields}$$

that occurred in the early 19-th century may be regarded as a sort of *distant ancestor* of the *transition*

Galois groups/abstract fields $\rightsquigarrow$ **abstract groups/anabelian algorithms**

that occurs in inter-universal Teichmüller theory [cf. the discussion at the beginning of §3.2 below; the discussion of §3.8 below; the discussion of the final portion of [Alien], §4.4, (i)].

A somewhat *more recent historical example* of this sort of situation may be seen in the situation surrounding the introduction of [possibly non-reduced] **schemes** by *Grothendieck* in the early 1960's. Indeed,

(RdVar) the possible existence of **nilpotent elements** in the structure sheaf of a *non-reduced scheme* struck many more classically oriented algebraic geometers, who were accustomed to working only with **reduced varieties** — whose geometry could be understood intuitively in terms of their sets of *closed points* — as being *entirely meaningless* and *unlikely to be of use in any substantive mathematical sense*, especially since it was taken as a *"matter of course"* or *"common sense"* that *any mathematically substantive property of a variety* would most certainly necessarily be *readily identifiable at the level of the set of closed points of the variety* [cf. the discussion of §1.5].

This more recent example of *non-reduced schemes* is especially of interest in the context of inter-universal Teichmüller theory in light of the *strong structural resemblances* that exist between the notion of **multiradiality** in inter-universal Teichmüller theory and the theory [due to Grothendieck!] of **crystals** [cf. [Alien], §3.1, (v); the discussion of (CrAND) in §3.5 below; the discussion of §3.10 below]. Indeed, the *"trivialization"* of the theory of crystals that results from replacing the *[non-reduced!] nilpotent thickenings* that appear in the theory of crystals by the associated reduced schemes corresponds *precisely* to the situation discussed in (CrRCS) [cf. also (CrOR)] in §3.5 below.

§**3.2.** <u>**RCS-redundancy of Frobenius-like/étale-like versions of objects**</u>

We begin by recalling that ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters — i.e., lattice points in the log-theta-lattice — give rise to both *Frobenius-like* and *étale-like* objects. Whereas the datum of a *Frobenius-like* object *depends*, *a priori*, on the coordinates "(n, m)" of the ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater from which it arises, *étale-like* objects satisfy various **[horizontal/vertical] coricity** properties to the effect that they map *isomorphically* to corresponding objects in a *vertically* [in the case of *vertical coricity*] or *horizontally* [in the case of *horizontal coricity*] neighboring ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater of the log-theta-lattice [cf. the discussion of Example 3.2.2, (i), (iv), below; [Alien], §2.7, (i), (ii), (iii), (iv); [Alien], §2.8, $2^{\mathrm{Fr}/\mathrm{\acute{e}t}}$; [Alien], §3.3, (ii), (vi), (vii); [Rpt2018], §15]. Here, we recall that *étale-like* objects correspond, for the most part, to

> **arithmetic fundamental groups** — such as, for instance, the *étale fundamental group* "$\pi_1(X)$" of a hyperbolic curve X over a number field or mixed characteristic local field

— or, more generally, to objects that may be *reconstructed* from such arithmetic fundamental groups, so long as the object is regarded as being *equipped with auxiliary data* consisting of the *arithmetic fundamental group* from which it was reconstructed, together with the *reconstruction algorithm* that was applied to reconstruct the object. Here, we recall that, in this context, it is of *fundamental importance* that these arithmetic fundamental groups be treated simply as *abstract topological groups* [cf. the discussion of §3.8 below for more details]. **Étale-like** objects also satisfy a

> *crucial* **symmetry** *property* with respect to **permutation** of **adjacent vertical lines** of the log-theta-lattice

[cf. Example 3.2.2, (ii), (iv), below; [Alien], §3.2; the discussion surrounding Fig. 3.12 in [Alien], §3.6, (i)]. That is to say, in summary,

> the crucial **coricity/symmetry** properties satisfied by **étale-like** objects — which are, in essence, a formal consequence of treating the arithmetic fundamental groups that appear as **abstract topological groups** [cf. the discussion of §3.8 below for more details] — play a central role in the **multiradial algorithms** of inter-universal Teichmüller theory [i.e., [IUTchIII], Theorem 3.11] and are **not** satisfied by **Frobenius-like** objects

— cf., the discussion of Example 3.2.2, (i), (ii), (iv), below; [Alien], §2.7, (ii), (iii); [Alien], §3.1, (iii); [Alien], Example 3.2.2; [Rpt2018], §15, (LbΘ), (Lb$\mathfrak{log}$), (LbMn), (EtFr), (EtΘ), (Et$\mathfrak{log}$), (EtMn).

On the other hand, once one implements the **RCS-identifications** discussed in (RC-$\mathfrak{log}$), (RC-Θ), there is, in effect, **"only one"** ($\Theta^{\pm\text{ell}}$NF-)Hodge theater in the log-theta-lattice, so all issues of determining relationships between corresponding objects in ($\Theta^{\pm\text{ell}}$NF-)Hodge theaters at *distinct coordinates "(n, m)"* of the log-theta-lattice appear, at first glance, to have been *"trivially resolved"*. Put another way,

> once one implements the **RCS-identifications** of (RC-$\mathfrak{log}$), (RC-Θ), even *Frobenius-like* objects appear, at first glance, to satisfy **all possible coricity/symmetry properties**, i.e., at a more symbolic level,
>
> $$(\text{RC-}\mathfrak{log}),\ (\text{RC-}\Theta) \quad \text{“} \Longrightarrow \text{”} \quad (\text{RC-FrÉt}).$$

In particular, the assertions of the RCS discussed in §3.1 and the present §3.2 may be summarized, at a symbolic level, as follows:

$$(\text{RC-}\mathfrak{log}),\ (\text{RC-}\Theta) \quad \text{“} \Longrightarrow \text{”} \quad (\text{RC-FrÉt}),\ (\text{RC-}\mathfrak{log}),\ (\text{RC-}\Theta) \quad \text{“} \Longrightarrow \text{”} \quad (\text{1-Dim}).$$

In fact, however, the RCS-identifications of (RC-$\mathfrak{log}$), (RC-Θ) do *not* resolve such issues [i.e., of relating corresponding objects in ($\Theta^{\pm\text{ell}}$NF-)Hodge theaters at *distinct coordinates "(n, m)"* of the log-theta-lattice] *at all* [cf. the discussion of *symmetries* in Example 2.3.1, (iii)!], but rather merely have the effect of

> *translating/reformulating* such issues of relating corresponding objects in ($\Theta^{\pm\text{ell}}$NF-)Hodge theaters at *distinct coordinates "(n, m)"* of the log-theta-lattice into issues of tracking the effect on objects in ($\Theta^{\pm\text{ell}}$NF-)Hodge theaters as one **moves along the paths** constituted by various composites of Θ- and $\mathfrak{log}$-links.

On the other hand, at a *purely formal level*,

> the discussion given above of the **falsity** of (RC-FrÉt) — i.e., as a consequence of the crucial **coricity/symmetry** properties discussed above — is, in some sense, predicated on the **falsity** of (RC-$\mathfrak{log}$), (RC-Θ).

This *falsity* of (RC-$\mathfrak{log}$), (RC-Θ) will be discussed in detail in §3.3, §3.4, below.

In this context, it is useful to observe that the situation surrounding the **Θ-link** and (RC-Θ), (RC-FrÉt) (respectively, the **$\mathfrak{log}$-link** and (RC-$\mathfrak{log}$), (RC-FrÉt)) is **structurally reminiscent** of the object $\mathbb{J}$ discussed in Examples 2.3.2, 2.4.1, 2.4.2 [cf. also the correspondences discussed in Example 2.4.5, (ii); the discussion of [IUTchIII], Remark 1.2.2, (vi), (vii)], i.e., if one regards

(StR1) the *domain* of the Θ- (respectively, $\mathfrak{log}$-) *link* as corresponding to ${}^{\dagger}\mathbb{I}$,

(StR2) the *codomain* of the Θ- (respectively, $\mathfrak{log}$-) *link* as corresponding to ${}^{\ddagger}\mathbb{I}$,

(StR3) the *gluing data* — i.e., a certain $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* (respectively, $\mathcal{F}$-*prime-strip*) — that arises from the *domain* ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater of the Θ- (respectively, $\mathfrak{log}$-) *link* as corresponding to ${}^{\dagger}\beta = \gamma_{\mathbb{J}}$,

(StR4) the *gluing data* — i.e., a certain $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* (respectively, $\mathcal{F}$-*prime-strip*) — that arises from the *codomain* ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater of the Θ- (respectively, $\mathfrak{log}$-) *link* as corresponding to $\gamma_{\mathbb{J}} = {}^{\ddagger}\alpha$,

(StR5) the *étale-like* objects that are *coric* with respect to the Θ- (respectively, $\mathfrak{log}$-) *link* as corresponding to the *glued differential* discussed in (AOD3), and

(StR6) the *RCS-identification* of (RC-Θ) (respectively, (RC-$\mathfrak{log}$)) as corresponding to the operation of passing to the quotient

$$\mathbb{J} \quad \twoheadrightarrow \quad \mathbb{M} \;=\; \mathbb{J}/\langle {}^{\dagger}\mathbb{I} \;\sim\; {}^{\ddagger}\mathbb{I}\rangle \quad \overset{\sim}{\rightarrow} \quad \mathbb{L} \;=\; \mathbb{I}/\langle \alpha \sim \beta\rangle.$$

This **strong structural similarity** will play an important role in the discussion of §3.3, §3.4, below.

Finally, we observe that the portion, i.e., (StR5), of this *strong structural similarity* involving the *glued differential* discussed in (AOD3) is particularly of interest in the context of the discussion of [Alien], §2. That is to say, as discussed in the first paragraph of [Alien], §2.6, **étale-like** objects in inter-universal Teichmüller theory play an analogous role to the role played by **tangent bundles/sheaves of differentials** in the

(InvHt) special case of the **invariance** of the **height under isogenies** between abelian varieties [due to Faltings] discussed in [Alien], §2.3, §2.4 [cf. also [Rpt2018], §16, (DiIsm), as well as the discussion of Example 3.2.1 below; the discussion of §3.5 below], as well as in the

(FrDff) discussion of **differentiation** of *p-adic liftings of the* **Frobenius morphism** given in [Alien], §2.5.

The *efficacy* of the technique of considering induced maps on *differentials* in the various examples discussed in [Alien], §2.3, §2.4, §2.5, may also be observed in the discussion of the *fundamental theorem of calculus* in §2.2, as well as in the context of (RC-FrÉt) and (StR5). In light of the *central importance* of (InvHt), as well as the closely related *coricity/symmetry/commutativity* properties of the log-theta-lattice [cf. the discussion at the beginning of the present §3.2, as well as the discussion at the beginning §3.3 below], in the *essential logical structure of inter-universal Teichmüller theory*, we pause to give a brief review/exposition of (InvHt) and these closely related coricity/symmetry/commutativity properties, in the following Examples 3.2.1, 3.2.2.

Example 3.2.1: Global multiplicative subspaces and bounds on heights.

(i) Let p be a *prime number*, K a *finite extension* of the field $\mathbb{Q}_p$ of p-adic numbers, E an *elliptic curve* over K with *bad multiplicative* — i.e., in other words, *nonsmooth semi-stable* — reduction over the *ring of integers* $\mathcal{O}_K$ of K. Write $\mathfrak{m}_K \subseteq \mathcal{O}_K$ for the maximal ideal of $\mathcal{O}_K$. Thus, E is a **Tate curve** and hence [cf. the theory of [Mumf2]] may be represented, using the theory of *formal schemes*, as a sort of *quotient*

$$\text{“}\mathbb{G}_{\mathrm{m}}/q_E^{\mathbb{Z}}\text{”}$$

of the *multiplicative group scheme* $\mathbb{G}_{\mathrm{m}}$ over K by the subgroup generated by the [nonzero] *q-parameter* $q_E \in \mathfrak{m}_K$ of the elliptic curve E. Let l be a prime number. Then the subscheme μ_l of l-torsion points of $\mathbb{G}_{\mathrm{m}}$ determines, via the above ***p*-adic quotient representation**, a *canonical exact sequence*

$$1 \longrightarrow \mu_l \longrightarrow E[l] \longrightarrow \mathbb{Z}/l\mathbb{Z} \longrightarrow 1$$

— where we write $E[l] \subseteq E$ for the subgroup scheme of l-torsion points of E, and we observe that the generator “q_E” of the group of deck transformations of the above quotient representation determines a *canonical generator* $\gamma_l \in (E[l]/\mu_l)(K)$, up to multiplication by ± 1, of $(E[l]/\mu_l)(K)$, i.e., a *canonical isomorphism*, up to multiplication by ± 1, of the quotient group scheme $E[l]/\mu_l$ with [the group scheme over K determined by] $\mathbb{Z}/l\mathbb{Z}$. Thus, in summary,

> the subgroup scheme $E[l] \subseteq E$ of l-torsion points of E is equipped with a canonical **multiplicative subspace** μ_l ($\hookrightarrow E[l]$), as well as with a **canonical generator**, up to multiplication by ± 1, of the quotient $E[l]/\mu_l$.

(ii) Let $(\mathbb{E}, 0_{\mathbb{E}})$ be the *pointed Riemann surface* determined by an *elliptic curve* over the field of *complex numbers* $\mathbb{C}$. Write $\mathbb{E}[\infty] \overset{\mathrm{def}}{=} \pi_1^{\mathrm{top}}(\mathbb{E})$ for the [usual topological] *fundamental group* of $\mathbb{E}$, relative to the basepoint determined by the origin $0_{\mathbb{E}}$ of the given elliptic curve. Thus, $\mathbb{E}[\infty]$ is a free abelian group on two generators. Let $\mathbb{M} \subseteq \mathbb{E}[\infty]$ be a rank one [free abelian] subgroup such that $\mathbb{E}[\infty]/\mathbb{M}$ is *torsion-free.* Then $\mathbb{M}$ corresponds to an *infinite covering*

$$\mathbb{E}_{\mathbb{M}} \to \mathbb{E}$$

of $\mathbb{E}$ such that any point $0_{\mathbb{E}_{\mathbb{M}}}$ of $\mathbb{E}_{\mathbb{M}}$ that lifts $0_{\mathbb{E}}$ determines, up to possible composition with the inversion automorphism, an isomorphism $\mathbb{E}_{\mathbb{M}} \overset{\sim}{\rightarrow} \mathbb{C}^{\times}$ of complex Lie groups, relative to the unique complex Lie group structure on the pointed Riemann surface $(\mathbb{E}_{\mathbb{M}}, 0_{\mathbb{E}_{\mathbb{M}}})$ that lifts the unique complex Lie group structure on the pointed Riemann surface $(\mathbb{E}, 0_{\mathbb{E}})$. In particular, we conclude that the choice of $\mathbb{M}$ [together with a choice of an isomorphism $\mathbb{E}_{\mathbb{M}} \overset{\sim}{\to} \mathbb{C}^{\times}$ of the sort just discussed] determines a **complex holomorphic quotient representation**

$$\text{“}\mathbb{C}^{\times}/q_{\mathbb{E}}^{\mathbb{Z}}\text{”}$$

of the given elliptic curve, together with an exact sequence

$$1 \longrightarrow \mathbb{M} \longrightarrow \mathbb{E}[\infty] \longrightarrow \mathbb{Z} \longrightarrow 1$$

— where we observe that the rank one free abelian group $\mathbb{E}[\infty]/\mathbb{M}$ ($\cong \mathbb{Z}$) is equipped with a *unique choice of generator* $\gamma_{\infty} \in \mathbb{E}[\infty]/\mathbb{M}$, up to multiplication by ±1. Thus, the *p-adic quotient representation*, as well as the associated *exact sequence* and *canonical generator* [up to multiplication by ±1], discussed in (i) may be regarded as *p-adic analogues* of the *complex holomorphic quotient representation* and associated *exact sequence/canonical generator* [up to multiplication by ±1] discussed in the present (ii).

(iii) Let F be a *number field* [i.e., a finite extension of the field $\mathbb{Q}$ of rational numbers] and E an *elliptic curve* over F. Write $E[l] \subseteq E$ for the subgroup scheme of l-torsion points of E. Then, in general,

> E does **not** necessarily admit a **global multiplicative subspace (GMS)** $M \subseteq E[l]$ or a **global canonical generator (GCG)** $\gamma \in (E[l]/M)(F)$, i.e., a subgroup scheme $M \subseteq E[l]$ or generator $\gamma \in (E[l]/M)(F)$ that *restricts* to the *multiplicative subspace* or *canonical generator* discussed in (i) at each nonarchimedean prime of F where E has *bad multiplicative reduction.*

On the other hand, let us *suppose*, for the remainder of the present (iii), that

> E *does admit* a **global multiplicative subspace (GMS)** $M \subseteq E[l]$.

Write $E^* \overset{\text{def}}{=} E/M$. Thus, we have an *isogeny*

$$\phi : E \to E^*$$

that, in light of the discussion of (i), together with the fact that $M \subseteq E[l]$ is a *GMS*, corresponds to the isogeny

$$\text{“}\mathbb{G}_{\mathrm{m}}/q_E^{\mathbb{Z}}\text{”} \longrightarrow \text{“}\mathbb{G}_{\mathrm{m}}/q_E^{l\cdot\mathbb{Z}}\text{”}$$

given by *raising to the l-th power* on $\mathbb{G}_{\mathrm{m}}$ at each nonarchimedean prime of F where E has *bad multiplicative reduction.* In particular, at each nonarchimedean prime v of F where E has *bad multiplicative reduction*, the *q-parameter* $q_{E^*,v}$ of E^* at v is the *l-th power* of the *q-parameter* $q_{E,v}$ of E at v, i.e.,

$$q_{E^*,v} = q_{E,v}^{l}.$$

In particular, if we write $\log(q_{(-)}) \in \mathbb{R}$ for the *normalized arithmetic degree* “$\underline{\deg}(-)$” [cf. the discussion preceding [GenEll], Definition 1.2] of the *arithmetic divisor* determined by the *q-parameters* of an elliptic curve over a number field at the nonarchimedean primes where the elliptic curve, which we denote "$(-)$", has *bad multiplicative reduction*, then we obtain the *relation*

$$\log(q_{E^*}) \quad = \quad l \cdot \log(q_E) \quad \in \mathbb{R}$$

between $\log(q_E), \log(q_{E^*}) \in \mathbb{R}$.

(iv) We continue to consider the situation discussed in (iii). Write

$$\overline{\mathcal{M}}_{\mathrm{ell}} \quad \supseteq \quad \mathcal{M}_{\mathrm{ell}}$$

for the *compactified moduli stack of elliptic curves* — or, equivalently, the moduli stack of *pointed stable curves of type* $(1,1)$ — over $\mathbb{Z}$ and the *open substack* obtained by forming the complement of the *divisor at infinity* $\infty_{\overline{\mathcal{M}}_{\mathrm{ell}}} \subseteq \overline{\mathcal{M}}_{\mathrm{ell}}$. Write $\omega_{\overline{\mathcal{M}}_{\mathrm{ell}}}$ for the **ample line bundle** on $\overline{\mathcal{M}}_{\mathrm{ell}}$ determined by the *cotangent space* at the origin of the *tautological family* of one-dimensional *semi-abelian schemes* over $\overline{\mathcal{M}}_{\mathrm{ell}}$ [i.e., obtained by forming the complement of the unique node of the tautological pointed stable curve of type $(1,1)$ over $\overline{\mathcal{M}}_{\mathrm{ell}}$]. Now recall the **discriminant moduli form** $\Delta_{\overline{\mathcal{M}}_{\mathrm{ell}}}$, which may be thought of as an *isomorphism* of line bundles

$$\mathcal{O}_{\overline{\mathcal{M}}_{\mathrm{ell}}} \quad \overset{\sim}{\to} \quad \omega^{\otimes 12}_{\overline{\mathcal{M}}_{\mathrm{ell}}}(-\infty_{\overline{\mathcal{M}}_{\mathrm{ell}}})$$

— i.e., a section of $\omega^{\otimes 12}_{\overline{\mathcal{M}}_{\mathrm{ell}}}$ over $\overline{\mathcal{M}}_{\mathrm{ell}}$ that has no zeroes or poles except for a zero of order 1 at $\infty_{\overline{\mathcal{M}}_{\mathrm{ell}}}$. It follows immediately from the existence of $\Delta_{\overline{\mathcal{M}}_{\mathrm{ell}}}$ [cf., e.g., the discussion of [GenEll], §3, for more details] that, if, for the sake of simplicity, we *ignore* the contributions at the *archimedean primes*, then we obtain the *relation*

$$\mathrm{ht}_{(-)} \quad \approx \quad \tfrac{1}{6}\log(q_{(-)})$$

— where we write $\mathrm{ht}(-)$ for the *normalized height* associated to the ample line bundle $\omega^{\otimes 2}_{\overline{\mathcal{M}}_{\mathrm{ell}}}$ on $\overline{\mathcal{M}}_{\mathrm{ell}}$ of the point determined by an elliptic curve "$(-)$" defined over a number field and "$\approx$" to signify a relationship of *bounded discrepancy* [i.e., that the absolute value of the difference between the left- and right-hand sides is bounded by some positive real number *independently* of "$(-)$"].

(v) We continue to consider the situation discussed in (iii), (iv). Recall the *isogeny* $\phi : E \to E^*$ discussed in (iii). Since this isogeny is of *degree* l, hence, in particular, *étale* over nonarchimedean primes of F of residue characteristic $\neq l$, we conclude immediately, via a *straightforward computation* of the map $d\phi$ induced on **differentials** by ϕ [cf. the proof of [GenEll], Lemma 3.5, for more details], that, if, for the sake of simplicity, we *ignore* the contributions at the *archimedean primes*, then we obtain relations

$$\mathrm{ht}_E - \log(l) \quad \lesssim \quad \mathrm{ht}_{E^*} \quad \lesssim \quad \mathrm{ht}_E + \log(l)$$

[where we use the notation "$\lesssim$" to signify an *inequality "$\leq$" up to bounded discrepancy*, i.e., a relation to the effect that the left-hand side is bounded, *independently* of the elliptic curve E and the prime number l, by the sum of the right-hand side and some positive real number] and hence, by combining the latter relation "$\lesssim$" with the relations of the final displays of (iii), (iv), that

$$\mathrm{ht}_E \quad \lesssim \quad \frac{1}{l}\left(\mathrm{ht}_E + \log(l)\right)$$

— i.e., that $\mathrm{ht}_E \lesssim \frac{1}{l-1}\log(l) \leq 1$. That is to say, in summary, if, for the sake of simplicity, we *ignore* the contributions at the *archimedean primes*, then

> the assumption [cf. (iii)] that E admits a **GMS** implies a **bound** on the **height** of E [i.e., ht_E] and hence, in particular, that, if one only considers number fields F of *bounded degree* over $\mathbb{Q}$, then there are **only finitely possibilities** for the isomorphism class of E.

This is precisely the argument given in [GenEll], Lemma 3.5, which may be regarded as a special case of the argument given in the original proof [due to Faltings] of the *invariance of the height* [up to bounded discrepancies] *under isogenies of abelian varieties.*

(vi) The argument reviewed in (v) may be understood as consisting of *two key points*, both of which are closely related to various *central aspects* of inter-universal Teichmüller theory. The *first key point* is, of course,

(vi-a) the assumption of the **existence** of a **GMS** [cf. (iii)], which implies that the passage $E \rightsquigarrow E^*$ to the quotient of E by the GMS corresponds to a *relation*

$$q_E \quad \mapsto \quad q_E^l$$

between the *q-parameters* of E and E^* at each nonarchimedean prime of F where E and E^* have *bad multiplicative reduction* — i.e., to a relation reminiscent of the **Frobenius morphism** in **positive characteristic**.

Here, we observe in passing that, in the *absence* of this crucial assumption of the *existence* of a *GMS*, the passage from E to some arbitrary quotient of E by a finite subgroup scheme of rank l would give rise to relations "$q_E \mapsto q_E^l$" at *some* nonarchimedean primes of bad multiplicative reduction and to relations "$q_E \mapsto q_E^{1/l}$" at *other* nonarchimedean primes of bad multiplicative reduction — i.e., a situation in which *the argument reviewed in (v) would break down completely*! In inter-universal Teichmüller theory,

(vi-b) the *fundamental role* played by **theta functions** [cf. the discussion of Examples 3.3.2, 3.8.4 below] — i.e., in the **multiradial reconstruction algorithms** of the **Θ-pilot** "$\{\underline{\underline{q}}_{v}^{j^2}\}$" — means that in addition to a **GMS**, it will be necessary to somehow **"simulate"** [cf. the discussion of Example 3.8.2, (i), below] the *existence* of a **GCG**.

Here, it is useful to recall that, from the point of view of the *classical complex theory of theta functions* [cf. also Example 3.3.2 below]

$$\sum_{n=-\infty}^{+\infty} q^{n^2} \cdot U^n$$

— where $q, U \in \mathbb{C}$ and $|q| < 1$:

- the "U" may be understood as the **standard multiplicative coordinate** on the *infinite covering* "$\mathbb{C}^\times \overset{\sim}{\to} \mathbb{E}_{\mathbb{M}} \to \mathbb{E}$" of (ii), hence is *only defined* once one has a *"multiplicative subspace* $\mathbb{M} \subseteq \mathbb{E}[\infty]$*"*, i.e., the *complex analogue* of the *l-torsion multiplicative subspace* "$\mu_l \subseteq E[l]$" of (i), or, alternatively, of the coverings "$\underline{X}_K$", "$\underline{C}_K$" of [Alien], §3.3, (i), (iv), (v) [cf. also the discussion of Example 3.8.2, (i), below];
- the "q" may be understood as the **complex *q*-parameter** determined by a *generating deck transformation* of the *finite covering* "$\mathbb{C}^\times \overset{\sim}{\to} \mathbb{E}_{\mathbb{M}} \to \mathbb{E}$" of (ii), hence is *only defined* once one has a *"generator, up to multiplication by* ± 1*, of* $\mathbb{E}[\infty]/\mathbb{M}$*"*, i.e., the *complex analogue* of the *l-torsion canonical generator* "$\gamma_l \in (E[l]/\mu_l)(K)$", up to multiplication by ± 1, or, alternatively, the index "$j = 1$" in the **Θ-pilot** "$\{\underline{\underline{q}}_{v}^{j^2}\}$" of [Alien], §3.3, (vii) [cf. also the discussion of Example 3.8.2, (i), below; the discussion of Example 3.8.4, (vi), below].

(vii) The *second key point* of the argument reviewed in (v)

(vii-a) consists of the *computation* of $d\phi$ discussed at the beginning of (v) — i.e., a computation that essentially amounts to the computation of the **logarithmic derivative**

$$d\log(U) = \tfrac{dU}{U} \quad \mapsto \quad l \cdot d\log(U)$$

of the *isogeny* ϕ, written as "$U \mapsto U^l$" in terms of the *standard multiplicative coordinate* "U" on $\mathbb{G}_{\rm m}$ [cf. (i), (iii)] — which, in light of the **ampleness** of $\omega_{\overline{\mathcal{M}}_{\rm ell}}$ [cf. (iv)], implies that

$$\text{“}\omega_{\overline{\mathcal{M}}_{\rm ell}}|_E \approx \omega_{\overline{\mathcal{M}}_{\rm ell}}|_{E^*}\text{”}$$

[i.e., the "roughly isomorphic" arithmetic line bundles obtained by restricting $\omega_{\overline{\mathcal{M}}_{\rm ell}}$ to E, E^*] serves, up to a *negligible discrepancy*, as a **common container** for the *moduli* of both E and E^*, i.e., in light of the existence of the **discriminant modular form** "$\Delta_{\overline{\mathcal{M}}_{\rm ell}}$" [cf. (iv)], as a *common container* for both "q_E" and "q_E^l".

Here, we observe that this *"common container"/ampleness* aspect of $\omega_{\overline{\mathcal{M}}_{\rm ell}}$ may be understood as corresponding — cf. the analogy between **étale-like** objects in inter-universal Teichmüller theory and **tangent bundles/sheaves of differentials** that was recalled above in the context of (InvHt), (FrDff)! — in inter-universal Teichmüller theory, to

(vii-b) the theory of the **$\mathfrak{log}$-link/log-shells** and closely related **mono-anabelian reconstruction algorithms** in a vertical line of the log-theta-lattice that give rise to the **$\mathfrak{log}$-Kummer-correspondence** of inter-universal Teichmüller theory, i.e., which play the *fundamental role* of furnishing a **multiradial container** for the **Frobenius-like Θ-pilot** at the lattice point $(0, 0)$ of the log-theta-lattice [cf. the discussion of §3.6, §3.9, §3.10, §3.11, below!].

Finally, in passing, we note that the **discriminant modular form** "$\Delta_{\overline{\mathcal{M}}_{\mathrm{ell}}}$" is also reminiscent of the **classical complex theta function** of Example 3.3.2 below [i.e., which may also be regarded as a *modular form* on the *upper half-plane*], as well as of the discussion of the relationship between the *discriminant modular form* and **scheme-theoretic Hodge-Arakelov theory** in the final portion of [HASurI], §1.2 [cf. also [Alien], Example 2.14.3; [Alien], §3.9, (i), (ii), for a discussion of the relationship between *scheme-theoretic Hodge-Arakelov theory* and inter-universal Teichmüller theory].

(viii) Thus, one may summarize the discussion of the present Example 3.2.1 as follows:

> At a very rough, introductory/expository level, one may think of **inter-universal Teichmüller theory** as a sort of *generalization* of the approach of (InvHt) [cf. (v)] to **bounding heights** of elliptic curves over number fields to the case of **[essentially] arbitrary** elliptic curves over number fields [i.e., which are *not* assumed to admit a GMS!] by
>
> - somehow **"simulating"** a **GMS/GCG** and
> - applying the theory of **theta functions** and **mono-anabelian geometry**.

Example 3.2.2: Coricity, symmetry, and commutativity properties of the log-theta-lattice. In the following discussion, we fix notation as follows: Let k be a *finite extension* of $\mathbb{Q}_p$, for some prime number p; $\overline{k}$ an *algebraic closure* of k; $q \in k$ a *nonzero element* of the maximal ideal $\mathfrak{m}_k$ of the ring of integers $\mathcal{O}_k$ of k; $N \geq 2$ an integer. Write

- $\mathbb{N}$ for the additive monoid of nonnegative integers;
- $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$;
- $\mathcal{O}_{\overline{k}}$ for the ring of integers of $\overline{k}$, with maximal ideal $\mathfrak{m}_{\overline{k}} \subseteq \mathcal{O}_{\overline{k}}$;
- $\mathcal{O}_{\overline{k}}^{\rhd} \subseteq \mathcal{O}_{\overline{k}}$ for the multiplicative monoid of nonzero elements of $\mathcal{O}_{\overline{k}}$;
- $\mathcal{O}_{\overline{k}}^{\times} \subseteq \mathcal{O}_{\overline{k}}^{\rhd}$ for the group of invertible elements of $\mathcal{O}_{\overline{k}}^{\rhd}$;
- $\mathcal{O}_{\overline{k}}^{\times} \twoheadrightarrow \mathcal{O}_{\overline{k}}^{\times\boldsymbol{\mu}}$, $\overline{k}^{\times} \twoheadrightarrow \overline{k}^{\times\boldsymbol{\mu}}$, $\overline{k} \twoheadrightarrow \overline{k}^{\boldsymbol{\mu}}$ for the respective quotients of $\mathcal{O}_{\overline{k}}^{\times}$, $\overline{k}^{\times\boldsymbol{\mu}}$, $\overline{k}^{\boldsymbol{\mu}}$ by the action of the group μ_{∞} of torsion elements [i.e., *roots of unity*] of $\mathcal{O}_{\overline{k}}^{\times}$;
- $\mathcal{F}_n \stackrel{\text{def}}{=} \mathcal{O}_{\overline{k}}^{\times\boldsymbol{\mu}} \times (q^n)^{\mathbb{N}} \subseteq \overline{k}^{\times\boldsymbol{\mu}}$ for the *multiplicative submonoid* of $\overline{k}^{\times\boldsymbol{\mu}}$ [equipped with a natural action by G_k] generated by $\mathcal{O}_{\overline{k}}^{\times\boldsymbol{\mu}}$ and q^n, where we allow n to be an arbitrary positive integer, and, by a slight abuse of notation, we write "q" for the image of $q \in \mathfrak{m}_k$ in $\overline{k}^{\times\boldsymbol{\mu}}$;
- $\Theta_{\overline{k}} : \mathcal{F}_N \overset{\sim}{\to} \mathcal{F}_1$ for the *isomorphism of* G_k*-monoids* that restricts to the *identity isomorphism* on $\mathcal{O}_{\overline{k}}^{\times\boldsymbol{\mu}}$ and maps $q^N \mapsto q$;
- $\log_{\overline{k}} : \mathcal{O}_{\overline{k}}^{\times} \twoheadrightarrow \overline{k}$ for the *p-adic logarithm* on $\mathcal{O}_{\overline{k}}^{\times}$.

Thus, G_k acts naturally on $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \twoheadleftarrow \mathcal{O}^{\times}_{\overline{k}} \subseteq \mathcal{O}^{\triangleright}_{\overline{k}} \subseteq \mathcal{O}_{\overline{k}}$. Let $\Pi \twoheadrightarrow G_k$ be a *topological group* equipped with a surjection onto G_k, which determines natural actions of Π on $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \twoheadleftarrow \mathcal{O}^{\times}_{\overline{k}} \subseteq \mathcal{O}^{\triangleright}_{\overline{k}} \subseteq \mathcal{O}_{\overline{k}}$.

(i) We begin by observing the following properties:

(i-a) The isomorphism

$$\Theta_{\overline{k}} : \mathcal{F}_N \xrightarrow{\sim} \mathcal{F}_1$$

is **not compatible** with the **ring structures** in its domain/codomain in the sense that it does **not** arise from a G_k*-equivariant ring homomorphism* $\phi : \overline{k} \hookrightarrow \overline{k}$, i.e., there does **not** exist a G_k*-equivariant ring homomorphism* $\phi : \overline{k} \hookrightarrow \overline{k}$ for which the induced map $\overline{k}^{\times\boldsymbol{\mu}} \to \overline{k}^{\times\boldsymbol{\mu}}$ restricts either to a map $\mathcal{F}_N \to \mathcal{F}_1$ that coincides with $\Theta_{\overline{k}}$ or to a map $\mathcal{F}_1 \to \mathcal{F}_N$ that coincides with $\Theta^{-1}_{\overline{k}}$.

(i-b) The map

$$\log_{\overline{k}} : \mathcal{O}^{\times}_{\overline{k}} \to \overline{k}$$

is **not compatible** with the **ring structures** in its domain/codomain in the sense that it does **not** arise from a G_k*-equivariant ring homomorphism* $\phi : \overline{k} \hookrightarrow \overline{k}$, i.e., there does **not** exist a G_k*-equivariant ring homomorphism* $\phi : \overline{k} \hookrightarrow \overline{k}$ that restricts to $\log_{\overline{k}}$.

Indeed, both (i-a) and (i-b) follow immediately from the easily verified elementary fact that any G_k*-equivariant ring homomorphism* $\phi : \overline{k} \hookrightarrow \overline{k}$ induces [by passing to G_k-invariants and considering the l*-divisibility* properties of units/non-units of k, for prime numbers $l \neq p$] an *isomorphism of topological fields* $k \xrightarrow{\sim} k$, hence an *isomorphism* between the *value groups* of the copies of k in the domain/codomain. [Here, we recall that, since $N \geq 2$, the assignments

$$q^N \mapsto q \quad \text{[cf. (i-a)]} \qquad \text{and} \qquad (\mathcal{O}^{\times}_{\overline{k}} \ni)\ 1 + p^2 \mapsto \log_{\overline{k}}(1 + p^2)\ (\in p \cdot \mathcal{O}_{\overline{k}}) \quad \text{[cf. (i-b)]}$$

yield *immediate contradictions* to the existence of such an *induced isomorphism* between *value groups*.] By contrast, we observe that

(i-c) the isomorphism $\Theta_{\overline{k}} : \mathcal{F}_N \xrightarrow{\sim} \mathcal{F}_1$ is **compatible** — in the sense of *equivariance*, relative to the natural actions on the domain/codomain of $\Theta_{\overline{k}}$ — with an **isomorphism** $G_k \xrightarrow{\sim} G_k$ between the copies of G_k in the domain/codomain of $\Theta_{\overline{k}}$;

(i-d) the map $\log_{\overline{k}} : \mathcal{O}^{\times}_{\overline{k}} \to \overline{k}$ is **compatible** — in the sense of *equivariance*, relative to the natural actions on the domain/codomain of $\log_{\overline{k}}$ — with an **isomorphism** $\Pi \xrightarrow{\sim} \Pi$ between the copies of Π in the domain/codomain of $\log_{\overline{k}}$.

Indeed, (i-c) and (i-d) follow immediately from the various definitions involved. Here, we recall, however, that it is of *fundamental importance* to observe the following:

(i-e) Since $\Theta_{\overline{k}}$ and $\log_{\overline{k}}$ are **not compatible** with the respective *ring structures* in their *domains/codomains* [cf. (i-a), (i-b)!], in order to obtain

coric structures — i.e., structures that are *commonly shared*, in an *invariant* fashion, by these domains/codomains, hence well-defined in a sense that is *independent* of any *specification of a relationship* to these domains/codomains — it is necessary to regard

- the *isomorphisms*

  $$G_k \xrightarrow{\sim} G_k \text{ and } \Pi \xrightarrow{\sim} \Pi$$

  as **indeterminate isomorphisms** of **abstract groups**, i.e., **not** of *Galois groups*, that is to say, groups equipped with the *"Galois-rigidification"* constituted by the auxiliary data of some sort of *action* on a *field/ring*;

- the *"identity isomorphism"* given by restricting $\Theta_{\overline{k}}$ to $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$

  $$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \xrightarrow{\sim} \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$$

  as an **indeterminate isomorphism** of **topological monoids** equipped with an action by a **topological group** [i.e., G_k], as well as with the collection of *submonoids* given by the images in $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$ of the intersections $\mathcal{O}^{\times}_{\overline{k}} \cap \overline{k}^H$ [i.e., of $\mathcal{O}^{\times}_{\overline{k}}$ with the H-invariants $\overline{k}^H \subseteq \overline{k}$ of $\overline{k}$], where H ranges over the open subgroups of G_k

— cf. the discussion of §3.8 below. These coricity properties will also play an important role in the context of the discussion of (ii-c) below.

In the context of (i-e), it is also important to note the following:

(i-f) The use of the terminology **"identity isomorphism"** when referring to any of the isomorphisms

$$G_k \xrightarrow{\sim} G_k, \quad \Pi \xrightarrow{\sim} \Pi, \quad \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \xrightarrow{\sim} \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$$

discussed in (i-e) can be *highly misleading* and give rise to *unnecessary confusion*, for the following reason: Strictly speaking, throughout mathematics, this terminology "identity isomorphism" is only well-defined when applied to an isomorphism from a mathematical object to *itself* [i.e., not to a *distinct* mathematical object!]. Since, however, the *ring structures* in the *domain/codomain* of $\Theta_{\overline{k}}$ or $\log_{\overline{k}}$ must be **distinguished** [i.e., so long as they are related to one another via $\Theta_{\overline{k}}$ or $\log_{\overline{k}}$ — cf. (i-a), (i-b), as well as the discussion of §3.4 below!], the *only possible "well-defined sense"* in which this terminology "identity isomorphism" may be applied is the sense of referring to some sort of "identity isomorphism" between **weaker underlying structures** [i.e., such as *"sets"*, *"abstract topological groups"*, *"abstract topological monoids"*, etc. — cf. the discussion of §3.8 below] that *do* indeed coincide, hence may indeed be *"identified"* with one another [cf. the situations discussed in Example 3.5.2 below].

The observations of (i-e) and (i-f) play a *fundamental role* in the *essential logical structure of inter-universal Teichmüller theory.*

(ii) Next, we observe the following properties:

(ii-a) The isomorphism

$$\Theta_{\overline{k}} : \mathcal{F}_N \overset{\sim}{\to} \mathcal{F}_1$$

is **not symmetric** with respect to switching the **domain/codomain** in the following sense: there do **not** exist G_k*-equivariant ring homomorphisms* $\phi : \overline{k} \hookrightarrow \overline{k}$, $\psi : \overline{k} \hookrightarrow \overline{k}$ such that the induced maps $\phi^{\times\boldsymbol{\mu}} : \overline{k}^{\times\boldsymbol{\mu}} \to \overline{k}^{\times\boldsymbol{\mu}}$, $\psi^{\times\boldsymbol{\mu}} : \overline{k}^{\times\boldsymbol{\mu}} \to \overline{k}^{\times\boldsymbol{\mu}}$ fit into a diagram

$$\begin{tikzcd}
\overline{k}^{\times\boldsymbol{\mu}} \arrow[d, "\phi^{\times\boldsymbol{\mu}}"] & \supseteq & \mathcal{F}_N \arrow[r, "\Theta_{\overline{k}}"] & \mathcal{F}_1 & \subseteq & \overline{k}^{\times\boldsymbol{\mu}} \arrow[d, "\psi^{\times\boldsymbol{\mu}}"] \\
\overline{k}^{\times\boldsymbol{\mu}} & \supseteq & \mathcal{F}_1 & \mathcal{F}_N \arrow[l, "\Theta_{\overline{k}}"'] & \subseteq & \overline{k}^{\times\boldsymbol{\mu}}
\end{tikzcd}$$

that is *commutative* on the *portion* of the *diagram* on which the relevant composites are defined.

(ii-b) The isomorphism

$$\log_{\overline{k}} : \mathcal{O}^{\times}_{\overline{k}} \overset{\sim}{\to} \overline{k}$$

is **not symmetric** with respect to switching the **domain/codomain** in the following sense: there do **not** exist G_k*-equivariant ring homomorphisms* $\phi : \overline{k} \hookrightarrow \overline{k}$, $\psi : \overline{k} \hookrightarrow \overline{k}$ that fit into a diagram

$$\begin{tikzcd}
\overline{k} \arrow[d, "\phi"] & \supseteq & \mathcal{O}^{\times}_{\overline{k}} \arrow[rr, "\log_{\overline{k}}"] & & \overline{k} \arrow[d, "\psi"] \\
\overline{k} & & \mathcal{O}^{\times}_{\overline{k}} \arrow[ll, "\log_{\overline{k}}"'] & \subseteq & \overline{k}
\end{tikzcd}$$

that is *commutative* on the *portion* of the *diagram* on which the relevant composites are defined.

Indeed, (ii-a) follows by tracing the image, at the level of *value groups*, of the q^N in the *upper left-hand corner* of the diagram, i.e., which maps [cf. the discussion of *induced isomorphisms of value groups* following (i-a), (i-b)] to q via the composite of the *upper horizontal* and *right-hand vertical* arrows, but to q^{N^2} ($\neq q^N$) via the composite of the *left-hand vertical* and *lower horizontal* arrows. In a similar vein, (ii-b) follows by tracing the image, at the level of *value groups*, of the element

$$u \overset{\mathrm{def}}{=} \{\exp(p^2)\}^{\frac{1}{p^2}} \in \mathcal{O}^{\times}_{\overline{k}}$$

[where "$\exp(-)$" denotes the well-known formal power series of the exponential function, and the "$\frac{1}{p^2}$" denotes a p^2-root of the element "$\{-\}$"] in the *upper left-hand corner* of the diagram, i.e., which maps to 0 via the composite of the *upper horizontal*, *right-hand vertical*, and *lower horizontal* arrows, but to u ($\neq 0$) via the *left-hand vertical* arrow. By contrast, we observe the following:

(ii-c) The *topological group actions* surrounding $\Theta_{\overline{k}}$ [cf. (i-c)]

$$\begin{tikzcd}
\Pi \arrow[r, two heads] & G_k \arrow[r, "\sim"] & G_k & \Pi \arrow[l, two heads] \\
& \curvearrowright & \curvearrowright & \\
& \mathcal{F}_N \arrow[r, "\Theta_{\overline{k}}"] & \mathcal{F}_1 &
\end{tikzcd}$$

yield a *diagram*

$$\Pi \twoheadrightarrow G_k \overset{\sim}{\rightarrow} G_k \twoheadleftarrow \Pi$$

that is [manifestly!] **symmetric** with respect to applying the operation of **reflection** of this last diagram around the " $\overset{\sim}{\rightarrow}$ ", where we observe that this *symmetry* property *only holds* if the **coricity** properties discussed in (i-e), (i-f) are applied, i.e., if

- the *isomorphism of topological groups* $G_k \overset{\sim}{\rightarrow} G_k$ is regarded as an **indeterminate isomorphism** of **abstract topological groups** and
- the arrows "$\twoheadrightarrow$"/"$\twoheadleftarrow$" are regarded as **surjections** between **abstract topological groups**, i.e., topological groups that are only well-defined up to *indeterminate isomorphism.*

Indeed, if one does *not* apply these coricity properties, then one is in effect working with structures [i.e., the various "Π" and "G_k" in "$\Pi \twoheadrightarrow G_k \overset{\sim}{\rightarrow} G_k \twoheadleftarrow \Pi$"] that **depend** on **Galois-rigidifications** that arise from structures *specific* to the *domain* or *codomain* of $\Theta_{\overline{k}}$, hence do **not** satisfy the desired *symmetry* property [cf. the discussion of (i-e), (i-f)!].

The observation (ii-c), together with the observations (i-e) and (i-f), plays a *fundamental role* in the *essential logical structure of inter-universal Teichmüller theory.*

(iii) Next, we observe the following properties:

(iii-a) $\Theta_{\overline{k}}$ does **not commute** with $\log_{\overline{k}}$ — i.e., at a *purely formal level*, "$\Theta_{\overline{k}} \circ \log_{\overline{k}} \neq \log_{\overline{k}} \circ \Theta_{\overline{k}}$" — in the following sense: there do **not** exist G_k*-equivariant ring homomorphisms* $\phi : \overline{k} \hookrightarrow \overline{k}$, $\psi : \overline{k} \hookrightarrow \overline{k}$ such that the induced maps $\phi^{\boldsymbol{\mu}} : \overline{k}^{\boldsymbol{\mu}} \to \overline{k}^{\boldsymbol{\mu}}$, $\psi^{\boldsymbol{\mu}} : \overline{k}^{\boldsymbol{\mu}} \to \overline{k}^{\boldsymbol{\mu}}$ fit into a diagram

$$\begin{tikzcd}
\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \arrow[d, "\log_{\overline{k}}"] & \subseteq \ \mathcal{F}_N \arrow[r, "\Theta_{\overline{k}}"] & \mathcal{F}_1 \ \supseteq & \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \arrow[d, "\log_{\overline{k}}"] \\
\overline{k} \arrow[d, "\phi^{\boldsymbol{\mu}}|_{\overline{k}}"] & & & \overline{k} \arrow[d, "\psi^{\boldsymbol{\mu}}|_{\overline{k}}"] \\
\overline{k}^{\boldsymbol{\mu}} & \supseteq \ \mathcal{F}_N \arrow[r, "\Theta_{\overline{k}}"] & \mathcal{F}_1 \ \subseteq & \overline{k}^{\boldsymbol{\mu}}
\end{tikzcd}$$

that is *commutative* on the *portion* of the *diagram* on which the relevant composites are defined.

(iii-b) $\Theta_{\overline{k}}$ is **not invariant** with respect to $\log_{\overline{k}}$ — i.e., at a *purely formal level*, "$\Theta_{\overline{k}} \circ \log_{\overline{k}} \neq \Theta_{\overline{k}}$" — in the following sense: there do **not** exist G_k-*equivariant ring homomorphisms* $\phi : \overline{k} \hookrightarrow \overline{k}$, $\psi : \overline{k} \hookrightarrow \overline{k}$ such that the induced maps $\phi^{\boldsymbol{\mu}} : \overline{k}^{\boldsymbol{\mu}} \to \overline{k}^{\boldsymbol{\mu}}$, $\psi^{\boldsymbol{\mu}} : \overline{k}^{\boldsymbol{\mu}} \to \overline{k}^{\boldsymbol{\mu}}$ fit into a diagram

$$\begin{tikzcd}
\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \arrow[d, "\log_{\overline{k}}"] & \subseteq \ \mathcal{F}_N \arrow[r, "\Theta_{\overline{k}}"] & \mathcal{F}_1 \ \supseteq & \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \arrow[d, "\iota_{\overline{k}}"] \\
\overline{k} \arrow[d, "\phi^{\boldsymbol{\mu}}|_{\overline{k}}"] & & & \overline{k}^{\boldsymbol{\mu}} \arrow[d, "\psi^{\boldsymbol{\mu}}"] \\
\overline{k}^{\boldsymbol{\mu}} & \supseteq \ \mathcal{F}_N \arrow[r, "\Theta_{\overline{k}}"] & \mathcal{F}_1 \ \subseteq & \overline{k}^{\boldsymbol{\mu}}
\end{tikzcd}$$

> — where $\iota_{\overline{k}} : \mathcal{O}_{\overline{k}}^{\times \boldsymbol{\mu}} \hookrightarrow \overline{k}^{\boldsymbol{\mu}}$ denotes the natural inclusion — that is *commutative* on the *portion* of the *diagram* on which the relevant composites are defined.

Indeed, let m be a positive integer such that $p^m \cdot \mathcal{O}_k \subseteq \log_{\overline{k}}(\mathcal{O}_k^{\times}) \subseteq p^{-m} \cdot \mathcal{O}_k$, where we write $\mathcal{O}_k^{\times} \overset{\mathrm{def}}{=} \mathcal{O}_k \cap \mathcal{O}_{\overline{k}}^{\times}$. Then (iii-a) follows by tracing the image of the element

$$u_n \overset{\mathrm{def}}{=} \exp(p^{n \cdot N}) \in \mathcal{O}_{\overline{k}}^{\times}$$

[where "exp(−)" denotes the well-known formal power series of the exponential function, and n is any positive integer $> 2m$ [which implies that $n < n + (n - 2m) = 2n - 2m \leq n \cdot N - 2m$] such that $p^n \in q^{\mathbb{N}} \cdot \mathcal{O}_{\overline{k}}^{\times}$] in the *upper left-hand portion* of the diagram, i.e., which maps [cf. the discussion of *induced isomorphisms of value groups* following (i-a), (i-b)] to [the image in the *value group* of k of] some nonzero element $\in p^{n \cdot N - 2m} \cdot \mathcal{O}_k$ [cf. the discussion of (i-e), as well as of Example 3.5.1 below] via the composite of the *upper horizontal* and *right-hand vertical* arrows, but to [the image in the *value group* of k of] p^n ($\notin p^{n \cdot N - 2m} \cdot \mathcal{O}_k$) via the *left-hand vertical* and *lower horizontal* arrows. In a similar vein, (iii-b) follows by tracing the image of the *same* element $u_n \in \mathcal{O}_{\overline{k}}^{\times}$ in the *upper left-hand portion* of the diagram, i.e., which maps [cf. the discussion of *induced isomorphisms of value groups* following (i-a), (i-b)] to [the image in the *value group* of k of] u_n via the composite of the *upper horizontal* and *right-hand vertical* arrows, but to [the image in the *value group* of k of] p^n ($\neq u_n$) via the *left-hand vertical* and *lower horizontal* arrows.

(iv) Finally, we observe that it is now an *essentially formal/routine* matter to *translate* the various elementary properties discussed above in (i), (ii), (iii) into the corresponding **coricity/symmetry/commutativity** properties of the *log-theta-lattice*, i.e.:

- the **incompatibility** with the **ring structures** in the **domain/codomain** [cf. the discussion at the beginning of the present §3.2; the discussion at the beginning of §3.8 below] of the **Θ-link** [cf. (i-a)] and **𝔩𝔬𝔤-link** [cf. (i-b)];
- the **horizontal** [cf. (i-c)] and **vertical** [cf. (i-d)] **coricity** [cf. (i-e), (i-f)] properties of the **étale-like structures** that appear in the log-theta-lattice [cf. the discussion at the beginning of the present §3.2; the discussion at the beginning of §3.8 below];
- the **non-symmetricity** with respect to switching the **domain/codomain** [cf. the discussion at the beginning of the present §3.2] of the **Θ-link** [cf. (ii-a)] and **𝔩𝔬𝔤-link** [cf. (ii-b)];
- the **symmetricity** with respect to the **Θ-link** [cf. the discussion at the beginning of the present §3.2] of the **étale-like structures** of the log-theta-lattice [cf. (ii-c)];
- the **non-commutativity** of the log-theta-lattice [cf. (iii-a); the discussion at the beginning §3.3 below];
- the **non-invariance** of the **Θ-link** with respect to the **𝔩𝔬𝔤-link** [cf. (iii-b); the discussion at the beginning §3.3 below].

§3.3. RCS-redundant copies in the domain/codomain of the $\mathfrak{log}$-link

The **Θ-link** of inter-universal Teichmüller theory is defined, in the style of *classical complex Teichmüller theory* [cf. Example 3.3.1 below; [IUTchI], Remark 3.9.3], as a **deformation** of the **ring structure** in a ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater that depends, in an essential way, on the **splitting** into **unit groups** and **value groups** of the various localizations of the number field involved. On the other hand, the **$\mathfrak{log}$-link** of inter-universal Teichmüller theory [i.e., in essence, the p-adic logarithm at primes of the number field of residue characteristic p] has the effect of **juggling/rotating** these unit groups and value groups, e.g., by mapping *units* to *non-units* [cf., e.g., the discussion of [Alien], Example 2.12.3, (v)]. In particular,

> there is **no natural way** to relate the *two Θ-links* [i.e., the *two horizontal arrows* in the following diagram] that emanate from the *domain* and *codomain* of the *$\mathfrak{log}$-link* [i.e., the *left-hand vertical arrow* in the following diagram]
>
> $$\begin{tikzcd} \bullet \arrow[r, "\Theta"] & \bullet \arrow[d, dash, dotted, "??"] \\ \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \bullet \end{tikzcd}$$

— that is to say, there is *no natural candidate* for "??" [i.e., such as, for instance, an *isomorphism* or the *$\mathfrak{log}$-link* between the two bullets "$\bullet$" on the *right-hand side* of the diagram] that makes the diagram *commute*. Indeed, it is an easy exercise [cf. Example 3.2.2, (iii), (iv); [Alien], §3.3, (ii); [Rpt2018], §15, (LbΘ), (Lb$\mathfrak{log}$), (LbMn)], to show that *neither* of these candidates for "??" [i.e., an *isomorphism* or the *$\mathfrak{log}$-link*] yields a commutative diagram.

Thus, in summary, any *identification* of the *domain* and *codomain* of the $\mathfrak{log}$-link [cf. (RC-$\mathfrak{log}$)!] yields a situation in which the *local splittings* into *unit groups* and *value groups* of the resulting identified "$\bullet$'s" are *no longer well-defined.* In particular,

> any such **identification** of the *domain* and *codomain* of the $\mathfrak{log}$-link [cf. (RC-$\mathfrak{log}$)!] yields a situation in which the **Θ-link** is **not well-defined**

— i.e., a situation in which *the apparatus of inter-universal Teichmüller theory completely ceases to function* — cf. the discussion of the *definition of the Θ-link* in the latter half of [Alien], §3.3, (ii), as well as the discussion of Example 3.3.3, (i), below. This discussion may be summarized, at a symbolic level, as follows:

$$\textit{definition} \text{ of the } \Theta\text{-link} \quad \Longrightarrow \quad \textit{falsity} \text{ of } (\text{RC-}\mathfrak{log}).$$

Next, we observe [cf. the discussion of [IUTchI], Remark 3.9.3, (iii), (iv)] that the *non-existence* of a *solution* for "??" in the above diagram [i.e., that makes the diagram *commute*] amounts, at a *structural* level, to essentially the *same* phenomenon as the **incompatibility** of the **dilations** that appear in *classical complex Teichmüller theory* with **multiplication** by **non-real roots of unity** [cf. Example 3.3.1 below]. Write $\mathbb{R}$, $\mathbb{C}$, respectively, for the topological fields of real and complex numbers. Then as observed in the discussion of the latter half of [Alien], §3.3, (ii) [cf., especially, the discussion surrounding [Alien], Fig. 3.6]:

$$\begin{tikzcd}
 & \vdots & \vdots & & & \vdots & \vdots \\
\cdots \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \cdots & & \bullet \arrow[u, "\mathfrak{log}"'] & \bullet \arrow[u, "\mathfrak{log}"'] \\
\cdots \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \cdots & \supseteq & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \\
\cdots \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \cdots & & \bullet \arrow[u, "\mathfrak{log}"'] & \bullet \arrow[u, "\mathfrak{log}"'] \\
 & \vdots \arrow[u, "\mathfrak{log}"'] & \vdots \arrow[u, "\mathfrak{log}"'] & & & \vdots \arrow[u, "\mathfrak{log}"'] & \vdots \arrow[u, "\mathfrak{log}"']
\end{tikzcd}$$

(InfH) this *structural similarity* is consistent with the *analogy* discussed in *loc. cit.* between

- the **"infinite H"** portion of the log-theta-lattice consisting of the *two vertical lines* [i.e., of $\mathfrak{log}$-links] on either side of a *horizontal arrow* [i.e., a Θ-link] of the log-theta-lattice and
- the elementary theory surrounding the **bijection**

$$\begin{array}{ccc} \mathbb{C}^\times \backslash GL_2^+(\mathbb{R})/\mathbb{C}^\times & \overset{\sim}{\to} & [0,1) \\ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) & \mapsto & \frac{\lambda-1}{\lambda+1} \end{array}$$

— where $\lambda \in \mathbb{R}_{\geq 1}$; $GL_2^+(\mathbb{R})$ denotes the group of 2×2 real matrices of positive determinant; $\mathbb{C}^\times$ denotes the multiplicative group of $\mathbb{C}$, which we regard as a subgroup of $GL_2^+(\mathbb{R})$ via the assignment $a + ib \mapsto \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$, for $a, b \in \mathbb{R}$ such that $(a, b) \neq (0, 0)$; the domain of the bijection is the set of double cosets.

That is to say,

- the **dilation** $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ — cf. the *dilations* that appear in classical complex Teichmüller theory, i.e., as reviewed in Example 3.3.1 below — corresponds to the **Θ-link** portion of an *"infinite H"* [cf. Example 3.3.2, (iii), below], while
- the two copies of the group of **toral rotations** "$\mathbb{C}^\times$" [e.g., by roots of unity in $\mathbb{C}^\times$] on either side of "$GL_2^+(\mathbb{R})$" — which may be thought of as a representation of the *holomorphic structures* in the domain and codomain of the *dilation* $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ [cf. the discussion of Example 3.3.1 below] — correspond, respectively, to the two *vertical lines* of **$\mathfrak{log}$-links** in the *"infinite H"* on either side of the Θ-link [cf. the discussion of Example 3.3.2, (iv), below].

Example 3.3.1: **Classical complex Teichmüller theory.** Let $\lambda \in \mathbb{R}_{>1}$. Recall the most *fundamental deformation of complex structure* in classical complex Teichmüller theory

$$\begin{array}{ccccccccc} & & \Lambda : \mathbb{C} & \to & \mathbb{C} & & & & \\ \mathbb{C} & \ni & z = x + iy & \mapsto & \zeta & = & \xi + i\eta \overset{\text{def}}{=} \lambda \cdot x + iy & \in & \mathbb{C} \end{array}$$

— where $x, y \in \mathbb{R}$. Let $n \geq 2$ be an integer, ω a *primitive* n*-th root of unity*. Write $(\omega \in)\ \mu_n \subseteq \mathbb{C}$ for the group of n-th roots of unity. Then *observe* that

> if $n \geq 3$, then there does *not* exist $\omega' \in \mu_n$ such that $\Lambda(\omega \cdot z) = \omega' \cdot \Lambda(z)$ for all $z \in \mathbb{C}$.

[Indeed, this *observation* follows immediately from the fact that if $n \geq 3$, then $\omega \notin \mathbb{R}$.] That is to say, in words,

> Λ is **not compatible** with multiplication by μ_n unless $n = 2$ [in which case $\omega = -1$].

This *incompatibility* with **"indeterminacies"** arising from multiplication by μ_n, for $n \geq 3$, may be understood as one fundamental reason for the *special role* played by **square differentials** [i.e., as opposed to n-th power differentials, for $n \geq 3$] in classical complex Teichmüller theory [cf. the discussion of [IUTchI], Remark 3.9.3, (iii), (iv)].

Example 3.3.2: **The Jacobi identity for the classical theta function.**

(i) Write $z = x + iy$ for the standard coordinate on the **upper half-plane** $\mathfrak{H} \overset{\text{def}}{=} \{z = x + iy \in \mathbb{C} \mid y > 0\}$. Recall the **theta function** on $\mathfrak{H}$

$$\Theta(q) \overset{\text{def}}{=} \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}n^2}.$$

— where we write $q \overset{\text{def}}{=} e^{2\pi i z}$. Restricting to the *imaginary axis* [i.e., $x = 0$] yields a function

$$\theta(t) \overset{\text{def}}{=} \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t}.$$

— where we write $t \overset{\text{def}}{=} y$.

(ii) Next, let us observe that

$$\iota \overset{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{C}^\times \subseteq GL^+(\mathbb{R})$$

maps $z \mapsto -z^{-1}$, hence $iy \mapsto iy^{-1}$, i.e., $t \mapsto t^{-1}$, while, for $\lambda \in \mathbb{R}_{\geq 1}$,

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{C}^\times \subseteq GL^+(\mathbb{R})$$

maps $z \mapsto \lambda \cdot z$, hence $iy \mapsto i\lambda \cdot y$, i.e., $t \mapsto \lambda \cdot t$.

(iii) Next, we observe the following:

- As $t \to +\infty$, the terms in the series for $\theta(t)$ are **rapidly decreasing**, and $\theta(t) \to +0$. In particular, the series for $\theta(t)$ is relatively **easy to compute**.
- As $t \to +0$, the terms in the series for $\theta(t)$ **decrease very slowly**, and $\theta(t) \to +\infty$. In particular, the series for $\theta(t)$ is **very difficult to compute**.

Thus, in summary, the *"flow/dilation"* $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ along the *imaginary axis* may be regarded as a sort of *"link"*, in the context of the **theta function** $\theta(t)$, between *small values* [i.e., $\theta(t) \to +0$ as $t \to +\infty$] and *large values* [i.e., $\theta(t) \to +\infty$ as $t \to +0$]. That is to say, this flow/dilation along the imaginary axis behaves in a way that is *strongly reminiscent* of the **Θ-link** of inter-universal Teichmüller theory [cf. the discussion of (InfH)].

(iv) The **Jacobi identity** for the **theta function** $\theta(t)$

$$\theta(t) \ = t^{-\frac{1}{2}} \cdot \theta(t^{-1})$$

allows one to analyze the behavior of $\theta(t)$ as $t \to +0$, which is *very difficult to compute* [cf. (iii)], in terms of the behavior of $\theta(t)$ as $t \to +\infty$, which is relatively *easy to compute* [cf. (iii)] — cf. the discussion of the *Jacobi identity* in [Pano], §3, §4; [Alien], §4.1, (i). Observe that this identity may be understood as a sort of **invariance** with respect to ι [cf. (ii)], up to a certain *easily computed factor* [i.e., $t^{-\frac{1}{2}}$]. Note that ι *"juggles"*, or *"rotates/permutes"*, the two dimensions of $\mathbb{R}^2$. This aspect of ι is strongly reminiscent of the **$\mathfrak{log}$-link** of inter-universal Teichmüller theory, which *"juggles"*, or *"rotates/permutes"*, the two underlying dimensions of the ring structures in a vertical column of the log-theta-lattice [cf., e.g., the discussion of [Alien], Example 2.12.3, (v)]. By contrast, we note that the theta function $\theta(t)$ does **not** satisfy any interesting properties of **invariance** with respect to the **dilations** $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

(v) Relative to the analogy with the **Θ-link** and **$\mathfrak{log}$-link** of inter-universal Teichmüller theory discussed in (iii), (iv), the *ι-invariance* interpretation of the Jacobi identity discussed in (iv) is *strongly reminiscent* of the central role played by **$\mathfrak{log}$-link invariance** in the construction of the **multiradial representation** in inter-universal Teichmüller theory [cf. the discussion surrounding ($\mathfrak{log}$ORInd), (Di/NDi) in §3.11 below; the discussion of the *Jacobi identity* in [Pano], §3, §4]. Here, we note that the *factor* $t^{-\frac{1}{2}}$ *in the Jacobi identity* may be understood as corresponding, relative to the analogy with inter-universal Teichmüller theory, to the *indeterminacies* (Ind1), (Ind2), (Ind3) acting on the *log-shells* in the multiradial representation. Indeed, both

- the **factor $t^{-\frac{1}{2}}$ in the Jacobi identity** — which amounts, in essence, to the well-known interpretation of the theta function $\theta(t)$ as a *modular form* on $\mathfrak{H}$ — and
- the **log-shells** in the multiradial representation — cf. the discussion of the relationship between *scheme-theoretic Hodge-Arakelov theory* and

> inter-universal Teichmüller theory in [Alien], Example 2.14.3; [Alien], §3.9, (i), (ii) —

are closely related to the notion of **"differentials"**.

(vi) At a more technical level, the crucial *ι-invariance* property of (iv), (v) may be understood as a consequence of the **Fourier transform invariance** of the **Gaussian** "$e^{-\Box^2}$" on the real line [cf. the discussion of the *Jacobi identity* in [Pano], §3, §4]. This *Fourier transform invariance* in turn may be understood as a consequence of the **quadratic form** "$\Box^2$" in the exponent of the Gaussian "$e^{-\Box^2}$", which may be thought of as the *first Chern class* of the ample line bundle whose section determines, via the canonical *theta trivialization* of the line bundle, the *theta function* under consideration [cf. the classical theory of complex theta functions as exposed, for instance, in [Mumf1], Chapter I]. When this quadratic form "$\Box^2$" is multiplied by a *factor* $t \in \mathbb{R}_{>0}$, application of the *Fourier transform* gives rise, up to suitable factors [involving, in particular, the **Gaussian integral**! — cf. the discussion of (v), as well as [Alien], §3.8], to the transformation

$$e^{-t\cdot\Box^2} \qquad \rightsquigarrow \qquad e^{-t^{-1}\cdot\Box^2}$$

that underlies the *Jacobi identity*. In this context, it is of central importance to observe that the transformation "$t \rightsquigarrow t^{-1}$" in the above display [i.e., as opposed to a transformation of the form "$t \rightsquigarrow c \cdot t^{-1}$", for some $c \in \mathbb{R}_{>0}$] is *indicative* of — and indeed in some sense essentially *equivalent* to — an **absolute notion of "1"** [i.e., the *unique invariant element* of the transformation $\mathbb{R}_{>0} \ni t \mapsto t^{-1} \in \mathbb{R}_{>0}$] in the copy of the real numbers that appears in the exponent "$-\Box^2$" of "$e^{-\Box^2}$". Finally, we observe that the *fundamental role* played by the quadratic form "$\Box^2$" of the above discussion in the proof of the *Fourier transform invariance* of the *Gaussian* that underlies the *Jacobi identity* is *strongly reminiscent* of the crucial **rigidity properties** in the theory of the *étale theta function* [cf. [Alien], §3.4, (iii), (iv); the discussion of the *Jacobi identity* in [Pano], §3, §4] that underlie the **multiradial representation** of inter-universal Teichmüller theory: Indeed, these rigidity properties may be understood as consequences of the **theta group symmetries**, which also arise, in essence, from the étale-theoretic version of the quadratic form "$\Box^2$" of the above discussion [cf. the discussion at the end of [Alien], §3.4, (iv)].

(vii) Before continuing our discussion, we pause briefly to make the following *elementary observation*:

> Let V be a 1-*dimensional* $\mathbb{R}$-*vector space.* Then a *[topological] ring/field structure* on V may be understood as a *multiplication map* $V \otimes_{\mathbb{R}} V \xrightarrow{\sim} V$ given by an isomorphism of $\mathbb{R}$-vector spaces. By tensoring with the dual vector space to V, one verifies immediately that such a multiplication map may be understood as an isomorphism of $\mathbb{R}$-vector spaces $V \xrightarrow{\sim} \mathbb{R}$, i.e., as the choice of a *nonzero element* in V given by the *image of* $1 \in \mathbb{R}$.

In particular, it follows immediately from this *elementary observation* that the **absolute notion of "$1 \in \mathbb{R}$"** discussed in (vi) may be interpreted as a [topological] **ring/field structure** on the copy of the real numbers that appears in the exponent

"$(-)$" of the Gaussian "$e^{(-)}$". This interpretation is *strongly reminiscent* of the central importance, in inter-universal Teichmüller theory, of working with the **first power** of [the reciprocal of the l-th root of] the **theta function** [cf., e.g., the discussion of [Alien], §3.4, (iii)], which makes it possible to consider the *truncated "mod N" Kummer theory* of the theta function: indeed, this **truncatibility** of the **Kummer theory** of the theta function is closely related to the [topological] **ring/field structure** of the local fields that appear in the context of the **$\mathfrak{log}$-link** of inter-universal Teichmüller theory [cf. the analogy between the $\mathfrak{log}$-*link* and "ι" discussed in (iv); the discussion of Example 3.8.4 below; the discussion of the final portion of [Alien], §3.6, (ii)].

(viii) Another important technical aspect of the *Fourier transform* discussed in (vi) is the *factor "e^{ixy}"* [where $x, y \in \mathbb{R}$] that appears in this Fourier transform. Indeed,

- this **"exponentiation of a complex unit** $\in \mathbb{S}^1 \subseteq \mathbb{C}^\times$**"** to a power given by some **indeterminate real number** — i.e., the real number that corresponds to the variable of integration in the Fourier ransform —

is *strongly reminiscent* of

- the **(Ind2) indeterminacy** action on the **local units** "$\mathcal{O}^{\times \boldsymbol{\mu}}$" in inter-universal Teichmüller theory — i.e., which amounts, in essence, to *exponentiation of these local units* to a power given by some **indeterminate element** $\in \widehat{\mathbb{Z}}^\times$

[cf. the discussion of the *Jacobi identity* in [Pano], §3, §4]. Moreover, in this context, it is of interest to note that the **integration** that occurs in the Fourier transform may be understood as corresponding to the **logical OR/XOR "$\vee/\dot{\vee}$"** aspect of the indeterminacies that occur in inter-universal Teichmüller theory — cf. the correspondence of *logical XOR "$\dot{\vee}$"* with *addition* [where we recall that *"integration"* may be understood as a sort of *"topological addition"* operation], as discussed in Example 2.4.6, (iii), as well as in ($\wedge(\dot{\vee})$-Chn) in §3.10 below.

Example 3.3.3: Theta functions and multiplicative structures.

(i) One *fundamental reason* for the *central role* played by **theta functions** in the *essential logical structure* of inter-universal Teichmüller theory lies in the fact that

(ThMlt) the *main properties of interest of the theta functions* that appear in inter-universal Teichmüller theory — most notably,

(ZrPl) the well-known description of the **zeroes/poles** of theta functions at the cusps [cf. [EtTh], Proposition 1.4, (i); [IUTchIII], Remark 2.3.3, (vi), (vii); [Alien], §3.4, (iii)], which yields — by applying the well-known intersection theory of divisors supported on the *special fiber* of the *universal topological covering* of the *Tate curve* [cf. the discussion preceding [EtTh], Proposition 1.1, of divisors supported on the *special fiber*] — a *"divisor-theoretic"* *characterization* of these theta functions [up to translation by a deck transformation of the universal topological covering];

(SymTh) the well-known **symmetries** of theta functions [cf. [EtTh], Proposition 1.4, (ii)];

(GalEv) the **Kummer-compatible Galois evaluation** properties of theta functions [cf. [IUTchII], Remark 1.12.4; the discussion at the beginning of [Alien], §3.6; [EtTh], Remark 1.10.4, (i)], which give rise to the *canonical splittings* of *theta monoids*, as well as to the construction of the *Gaussian monoids* [cf. [IUTchII], Corollaries 2.5, 2.6, 3.5, 3.6; [Alien], §3.4, (iii); [Alien], §3.6, (ii)]

— may be **expressed** *entirely in terms of the* **multiplicative** *structures of the various rings* that appear, i.e., *without invoking the additive structures* of these rings.

This *expressibility purely in terms of multiplicative structures* plays an *essential role* in establishing the

· **multiradial unit group/value group splittings/decouplings** [cf. [IUTchII], Remark 1.12.2, (vi); [Alien], §3.4, (iii)] and
· **non-interference** properties [cf. [Alien], §3.7, (i)]

that underlie the *definition* of the **Θ-link** [cf. the discussion of the present §3.3 preceding (InfH)] and **$\mathfrak{log}$-Kummer-correspondence** [cf. [IUTchII], Remark 1.12.2, (iv); [IUTchIII], Remark 1.2.3].

(ii) In the context of (ZrPl), it is important to recall the *central importance* of the fact that

(ZrPlOrd) the *signed order* [i.e., "+" for *zeroes*, "−" for *poles*] of the theta function at each of the cusps is **precisely one** [cf. [IUTchIII], Remark 2.3.3, (vi), (vii); [Alien], §3.4, (iii)].

This property (ZrPlOrd) of the theta functions that appear in inter-universal Teichmüller theory is closely related to the properties discussed in Example 3.3.2, (vi), (vii), in the case of *complex theta functions.* In inter-universal Teichmüller theory, this property (ZrPlOrd) ensures that the **mono-theta-theoretic cyclotomic rigidity** algorithms that arise from the theory of *theta functions* are

· **compatible** with the **topology** of the **tempered fundamental group** and, moreover, are
· **not subject** to $\{\pm 1\}$-*indeterminacies*

— properties that are *not* satisfied by the cyclotomic rigidity isomorphisms that arise from the theory of *algebraic rational functions* [cf. [IUTchIII], Remark 2.3.3, (vi), (vii); [IUTchIII], Remark 3.11.4; [Alien], §3.4, (iii), as well as the discussion of Examples 3.3.2, (vi), (vii); 3.8.4, (iv), (v), (vi), below, of the present paper]. Here, it is of interest to recall that

(MltAdd) the *anabelian reconstruction algorithms* of [AbsTopIII], Theorem 1.9, imply that the *additive structure* of the function field of an algebraic curve may in fact be *reconstructed* from the *multiplicative structure* of the function field of algebraic rational functions on the curve, equipped with the

> [*multiplicative*!] valuation maps and [*multiplicative*!] evaluation maps at the closed points of the curve [cf. *"Uchida's Lemma"*, i.e., [AbsTopIII], Proposition 1.3].

That is to say, (MltAdd) means that once one allows oneself to work with *algebraic rational functions* — i.e., as opposed to *theta functions* — the issue emphasized in (i) of **expressibility purely in terms of multiplicative structures** in some sense *ceases to be well-defined/non-vacuous.* By contrast, if one *restricts* oneself, as is indeed the case in the discussion of (i), to considering *theta functions* — as is necessary, in order to apply the *essential property (ZrPlOrd)* discussed above! — then (MltAdd) is *no longer applicable*, so there are *no longer any obstructions to the "well-definedness/non-vacuousness"* of the notion of *"expressibility purely in terms of multiplicative structures"*.

Finally, we recall that, in any *vertical line* of $\mathfrak{log}$-links in the log-theta-lattice,

- the *discrepancy* between the *[holomorphic] Frobenius-like* copies of objects on either side of a $\mathfrak{log}$-link [cf. (RC-$\mathfrak{log}$)], as well as
- the *discrepancy* between *[holomorphic] Frobenius-like* copies of objects and *[holomorphic] étale-like* copies of objects [cf. (RC-FrÉt)],

may be understood as the extent to which the diagram of arrows that constitutes the **$\mathfrak{log}$-Kummer-correspondence** associated to this vertical line of $\mathfrak{log}$-links **fails to commute**.

This failure to commute may be *estimated* by means of the **indeterminacy (Ind3)**, i.e., by interpreting this failure to commute as a sort of **"upper semi-commutativity"**. This indeterminacy (Ind3) is *highly nontrivial* and, in particular, gives rise to the **inequality** that appears in the final computation of log-volumes in inter-universal Teichmüller theory [cf. [IUTchIII], Corollary 3.12]. In this context, it is important to recall that the theory surrounding this indeterminacy (Ind3) depends, in an essential way, on the *absolute anabelian geometry* of [AbsTopIII], §1, i.e., which allows one to reconstruct a hyperbolic curve X over a number field or mixed characteristic local field from the *abstract profinite group* determined by the étale fundamental group $\pi_1(X)$ of the curve. That is to say, in summary, this *absolute anabelian geometry* allows one to show that

> the **discrepancies** between the various *[holomorphic] Frobenius-like* and *[holomorphic] étale-like* copies of objects in a vertical line of $\mathfrak{log}$-links [cf. (RC-$\mathfrak{log}$), (RC-FrÉt)] in the log-theta-lattice are **"bounded by"** the [relatively mild] indeterminacy (Ind3).

On the other hand, this *absolute anabelian geometry* most certainly does *not* imply that these discrepancies are trivial/non-existent, i.e., as asserted in (RC-$\mathfrak{log}$), (RC-FrÉt) — cf. the discussion of the **falsity** of (RC-$\mathfrak{log}$), (RC-FrÉt) in §3.2 and the present §3.3.

§3.4. RCS-redundant copies in the domain/codomain of the Θ-link

The **Θ-link** of inter-universal Teichmüller theory

$$\bullet \xrightarrow{\Theta} \bullet$$

is defined as a **gluing** between the ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater "$\bullet$" in the *domain* of the arrow and the ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater "$\bullet$" in the *codomain* of the arrow along $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* "$*$" that arise from the **$\Theta$-pilot object** "$\Theta$-$\mathfrak{plt}$" in the *domain* and the ***q*-pilot object** "$\mathfrak{q}$-$\mathfrak{plt}$" in the *codomain*. Here, it is important to note that this gluing is obtained by regarding these $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips "$*$" as being known *only up to isomorphism*. This point of view, i.e., of regarding these $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips "$*$" as being known only up to isomorphism, is implemented formally by taking the gluing to be the *full poly-isomorphism* — i.e., the set of all isomorphisms — between the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips arising from the *domain* and *codomain* of the Θ-link. Here, we recall that

- $\mathfrak{q}$-$\mathfrak{plt}$ essentially amounts to the arithmetic line bundle determined by [the ideal generated by] some $2l$-th root $\underline{\underline{q}}_{\underline{v}}$ of the q-parameter at the valuations $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, while
- Θ-$\mathfrak{plt}$ essentially amounts to the collection of arithmetic line bundles determined by [the ideals generated by] the collection $\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}$, as j ranges over the integers $1, \dots, l^{*} \stackrel{\mathrm{def}}{=} \frac{l-1}{2}$ [where l is the prime number that appears in the *initial* Θ*-data* under consideration], and $\underline{v}$ ranges over the valuations $\in \underline{\mathbb{V}}^{\mathrm{bad}}$.

Also, we recall that each ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater "$\bullet$" gives rise to an associated model "$\mathfrak{Ring}$" of the ring/scheme theory surrounding the elliptic curve under consideration. In the following discussion, we shall write

- ${}^{\dagger}\bullet$ for the "$\bullet$" in the *domain* of the Θ-link,
- ${}^{\ddagger}\bullet$ for the "$\bullet$" in the *codomain* of the Θ-link,
- $\square$ for an *arbitrary element* of the set consisting of "$\dagger$", "$\ddagger$", and the "*empty symbol*" [i.e., no symbol at all],
- ${}^{\square}\Theta$-$\mathfrak{plt} \in {}^{\square}\mathfrak{Ring}$ for the Θ-pilot arising from the collection "$\{{}^{\square}\underline{\underline{q}}_{\underline{v}}^{j^2}\}$" that appears in the model of ring/scheme theory associated to ${}^{\square}\bullet$, and
- ${}^{\square}\mathfrak{q}$-$\mathfrak{plt} \in {}^{\square}\mathfrak{Ring}$ for the q-pilot arising from the "${}^{\square}\underline{\underline{q}}_{\underline{v}}$" that appears in the model of ring/scheme theory associated to ${}^{\square}\bullet$.

Finally, we recall that since, for $j \neq 1$, the *valuation* [at each valuation $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] of $\underline{\underline{q}}_{\underline{v}}^{j^2}$ *differs* from that of $\underline{\underline{q}}_{\underline{v}}$, the arithmetic degrees of the line bundles constituted by $\mathfrak{q}$-$\mathfrak{plt}$ and Θ-$\mathfrak{plt}$ *differ*.

Thus, at a more formal level, the above description of the *gluing* that constitutes the Θ-link may be summarized as follows:

$${}^{\dagger}\mathfrak{Ring} \ni {}^{\dagger}\Theta\text{-}\mathfrak{plt} \;\; \leftarrow: \;\; * \;\; :\rightarrow \;\; {}^{\ddagger}\mathfrak{q}\text{-}\mathfrak{plt} \in {}^{\ddagger}\mathfrak{Ring}$$

$$\mathfrak{Ring} \ni \mathfrak{q}\text{-}\mathfrak{plt} \neq \Theta\text{-}\mathfrak{plt} \in \mathfrak{Ring}$$

[where "$\leftarrow:$" and "$:\rightarrow$" denote the assignments that consitute the *gluing* discussed above].

In this context, we note the following **fundamental observation**, which underlies the **entire logical structure** of inter-universal Teichmüller theory [cf. the discussion of [IUTchIII], Remark 3.12.2, ($\mathrm{c}^{\mathrm{itw}}$), ($\mathrm{f}^{\mathrm{itw}}$); [Alien], §3.11, (iv)]:

(AOΘ1) the following condition **holds**:

$$\Big(* \ :\to \ {}^{\dagger}\Theta\text{-}\mathfrak{plt} \in {}^{\dagger}\mathfrak{Ring} \Big) \ \wedge \ \Big(* \ :\to \ {}^{\ddagger}\mathfrak{q}\text{-}\mathfrak{plt} \in {}^{\ddagger}\mathfrak{Ring} \Big).$$

By contrast, if one simply **deletes** the **distinct labels** "†", "‡" [cf. (RC-Θ)!], then

(AOΘ2) the following condition **holds**:

$$\Big(* \ :\to \ \Theta\text{-}\mathfrak{plt} \in \mathfrak{Ring} \Big) \ \vee \ \Big(* \ :\to \ \mathfrak{q}\text{-}\mathfrak{plt} \in \mathfrak{Ring} \Big).$$

Of course,

(AOΘ3) the essential mathematical content discussed in this condition (AOΘ2) may be *formally* described as a condition involving the *AND relator* "$\wedge$":

$$\Big(\mathfrak{q}\text{-}\mathfrak{plt} \in \{\mathfrak{q}\text{-}\mathfrak{plt}, \Theta\text{-}\mathfrak{plt}\} \Big) \quad \wedge \quad \Big(\Theta\text{-}\mathfrak{plt} \in \{\mathfrak{q}\text{-}\mathfrak{plt}, \Theta\text{-}\mathfrak{plt}\} \Big).$$

On the other hand, precisely as a *consequence* of the fact [discussed above] that $\mathfrak{Ring} \ni \mathfrak{q}\text{-}\mathfrak{plt} \neq \Theta\text{-}\mathfrak{plt} \in \mathfrak{Ring}$,

(AOΘ4) the following condition does **not** hold:

$$\Big(* \ :\to \ \Theta\text{-}\mathfrak{plt} \in \mathfrak{Ring} \Big) \ \wedge \ \Big(* \ :\to \ \mathfrak{q}\text{-}\mathfrak{plt} \in \mathfrak{Ring} \Big).$$

That is to say, the operation of **identifying** ${}^{\dagger}\bullet$, ${}^{\ddagger}\bullet$ [hence also ${}^{\dagger}\mathfrak{Ring}$, ${}^{\ddagger}\mathfrak{Ring}$] — e.g., on the grounds of **"redundancy"** [i.e., as asserted in (RC-Θ)!] — by **deleting** the **distinct labels** "†", "‡" has the effect of passing from a situation in which

the **AND relator** *"$\wedge$" holds [cf. (AOΘ1)]*

to a situation in which

the **OR relator** *"$\vee$" holds [cf. (AOΘ2), (AOΘ3)],* but
the **AND relator** *"$\wedge$" does* **not** *hold [cf. (AOΘ4)]!*

In particular, relative to the *correspondences*

$$ {}^{\dagger}\bullet, {}^{\dagger}\mathfrak{Ring} \quad \longleftrightarrow \quad {}^{\dagger}\mathbb{I}; \qquad * \quad \longleftrightarrow \quad \gamma_{\mathbb{J}}; \qquad {}^{\ddagger}\bullet, {}^{\ddagger}\mathfrak{Ring} \quad \longleftrightarrow \quad {}^{\ddagger}\mathbb{I}$$

$$ {}^{\dagger}\Theta\text{-}\mathfrak{plt} \quad \longleftrightarrow \quad {}^{\dagger}\beta; \qquad {}^{\ddagger}\mathfrak{q}\text{-}\mathfrak{plt} \quad \longleftrightarrow \quad {}^{\ddagger}\alpha$$

[cf. the correspondences (StR1) $\sim$ (StR6) discussed in §3.2; the correspondences discussed in Example 2.4.5, (ii); the discussion of [Alien], §3.11, (iv)], one obtains very precise **structural resemblances**

$$\begin{array}{ccc} (AO\Theta 1) & \longleftrightarrow & (AOL1), \\ (AO\Theta 2) & \longleftrightarrow & (AOL2), \\ (AO\Theta 3) & \longleftrightarrow & (AOL3), \\ (AO\Theta 4) & \longleftrightarrow & (AOL4) \end{array}$$

with the situation discussed in Example 2.4.1, (i), (ii). Thus, in summary,

> the **falsity** of (RC-Θ) may be understood as a consequence of the **falsity** [cf. (AOΘ4)] of the crucial **AND relator** "$\wedge$" in the **absence** of **distinct labels**, in stark contrast to the **truth** [cf. (AOΘ1)] of the crucial **AND relator** "$\wedge$" as an essentially *tautological consequence* of the use of the **distinct labels** "†", "‡".

In the context of the *central role* played in the logical structure of inter-universal Teichmüller theory by the **validity** of (AOΘ1), it is important to note [cf. the property discussed in (AOΘ4)!] that

(NoRng) there does **not** exist an isomorphism of **ring structures** ${}^{\dagger}\mathfrak{Ring} \overset{\sim}{\rightarrow} {}^{\ddagger}\mathfrak{Ring}$ that induces, on value groups of corresponding local rings, the *desired assignment* $\{{}^{\dagger}\underline{\underline{q}}_{\underline{v}}^{j^2}\} \mapsto {}^{\ddagger}\underline{\underline{q}}_{\underline{v}}$ [i.e., that appears in the Θ-link].

On the other hand, if, instead of considering the *full ring structures* of ${}^{\dagger}\mathfrak{Ring}$, ${}^{\ddagger}\mathfrak{Ring}$, one considers [cf. the discussion of [Rpt2018], §6]

- certain suitable subquotients — i.e., in the notation of [Alien], §3.3, (vii), (a^q), (a^{Θ}), "$\mathcal{O}^{\times}_{\overline{k}}$" — of the **underlying multiplicative monoids** of corresponding local fields, as well as
- the absolute Galois groups — i.e., in the notation of [Alien], §3.3, (vii), (a^q), (a^{Θ}), "G_k" — associated to corresponding local rings, regarded as **abstract topological groups** [that is to say, **not** as Galois groups, or equivalently/alternatively, as groups of field automorphisms! — cf. the discussion of §3.8 below],

then one obtains structures — i.e., the structures that constitute the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strips* that appear in the Θ-link — that are **simultaneously associated** [as "underlying structures"] to *both* ${}^{\dagger}\mathfrak{Ring}$ *and* ${}^{\ddagger}\mathfrak{Ring}$ via *isomorphisms* [i.e., of certain suitable *multiplicative monoids* equipped with actions by certain suitable *abstract topological groups*] that restrict, on the subquotient monoids that correspond to the respective value groups, to the *desired assignment* $\{{}^{\dagger}\underline{\underline{q}}_{\underline{v}}^{j^2}\} \mapsto {}^{\ddagger}\underline{\underline{q}}_{\underline{v}}$. It is this *crucial simultaneity* that yields, as a tautological consequence, the **validity** of the **AND relator** "$\wedge$" in (AOΘ1).

Working, as in the discussion above, with *multiplicative monoids* equipped with actions by *abstract topological groups*, necessarily gives rise to certain **indeterminacies**, called (Ind1), (Ind2), that play an important role in inter-universal Teichmüller theory. Certain aspects of these indeterminacies (Ind1), (Ind2) will be discussed in more detail in §3.5 below. In this context, we recall that one central assertion of the RCS [cf. the discussion of Example 3.2.2; the discussion of (SSInd), (SSId) in [Rpt2018], §7, §10] is to the effect that

(NeuRng) these indeterminacies (Ind1), (Ind2) may be *eliminated*, without affecting the essential logical structure of inter-universal Teichmüller theory, by taking the *multiplicative monoids* and *abstract topological groups* that appear in the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips of the above discussion to be equipped with **rigidifications** by regarding them as arising from some **fixed "neutral" ring structure** ${}^{\boxdot}\mathfrak{Ring}$.

On the other hand, as discussed in (NoRng) above, there does **not** exist any ring structure that is *compatible* [i.e., in the sense discussed in (NoRng)], with the desired assignment $\{{}^{\dagger}\underline{\underline{q}}_{\underline{v}}^{j^2}\} \mapsto {}^{\ddagger}\underline{\underline{q}}_{\underline{v}}$. That is to say, in summary,

(NeuORInd) working with such a *fixed "neutral" ring structure* ${}^{\boxdot}\mathfrak{Ring}$ as in (NeuRng) means *either* that

(NeuORInd1) there is **no relationship** between "$*$" and ${}^{\boxdot}\mathfrak{Ring}$, *or* that

(NeuORInd2) the relationship between "$*$" and ${}^{\boxdot}\mathfrak{Ring}$ is *always necessarily subject* to an **indeterminacy** [cf. (AOΘ2), (AOΘ3)!]

$$\left(* \ :\to \ {}^{\boxdot}\Theta\text{-}\mathfrak{plt} \in {}^{\boxdot}\mathfrak{Ring} \right) \ \vee \ \left(* \ :\to \ {}^{\boxdot}\mathfrak{q}\text{-}\mathfrak{plt} \in {}^{\boxdot}\mathfrak{Ring} \right)$$

[cf. the situation discussed in Example 3.2.2; the situation discussed in [Rpt2018], §10, (SSId)].

Here, we observe that whichever of these "options"/"indeterminacies" that appear in (NeuORInd) [i.e., (NeuORInd1), (NeuORInd2)] one chooses to adopt, one is forced to contend with an indeterminacy that is, in some sense, *much more drastic* than the relatively mild indeterminacies (Ind1), (Ind2) whose *elimination* formed the *original motivation* for the introduction of ${}^{\boxdot}\mathfrak{Ring}$!

Finally, we *observe* that this *much more drastic indeterminacy* (NeuORInd) means [cf. the discussion of Example 2.4.4!] that throughout any argument, one must always take the position that the *only possible relationship* between "$*$" and ${}^{\boxdot}\Theta\text{-}\mathfrak{plt}$, ${}^{\boxdot}\mathfrak{q}\text{-}\mathfrak{plt}$ is one in which

(PltRel) *"$*$" maps either to* ${}^{\boxdot}\Theta\text{-}\mathfrak{plt}$ *or — i.e., "$\vee$"! — to* ${}^{\boxdot}\mathfrak{q}\text{-}\mathfrak{plt}$*, but* **not both***!*

Since ${}^{\ddagger}\mathfrak{Ring}$ may be thought of as a ring structure in which "$*$" *tautologically* maps to ${}^{\ddagger}\Theta\text{-}\mathfrak{plt}$, while ${}^{\dagger}\mathfrak{Ring}$ may be thought of as a ring structure in which "$*$" *tautologically* maps to ${}^{\ddagger}\mathfrak{q}\text{-}\mathfrak{plt}$, one may rephrase the above *observation* as the observation that one must always take the position that the *only possible relationship* between ${}^{\boxdot}\mathfrak{Ring}$, on the one hand, and ${}^{\ddagger}\mathfrak{Ring}$, ${}^{\dagger}\mathfrak{Ring}$, on the other, is one in which

(RngRel) *the ring structure* ${}^{\boxdot}\mathfrak{Ring}$ *is identified either with the ring structure* ${}^{\ddagger}\mathfrak{Ring}$ *or — i.e., "$\vee$"! — with the ring structure* ${}^{\dagger}\mathfrak{Ring}$*, but* **not both***!*

At this point, let us recall [cf., e.g., the discussion of §3.5, §3.11, below; [Rpt2018], §9, (GIUT), (ΘCR)] that

> inter-universal Teichmüller theory requires, in an essential way, the use of the $\mathfrak{log}$**-links**, hence, in particular, [in order to define the *power series* of the various p-adic logarithm functions that constitute these $\mathfrak{log}$-links!] the *ring structures* ${}^{\dagger}\mathfrak{Ring}$, ${}^{\ddagger}\mathfrak{Ring}$ on **both sides** — i.e., "$\wedge$"! — of the Θ-link

[cf. the discussion surrounding (InfH) of the two *vertical lines* of $\mathfrak{log}$*-links* in the *"infinite H"* on either side of the Θ-link]. In particular, we conclude formally that

> *it is impossible to implement the arguments of inter-universal Teichmüller theory once this sort of* **much more drastic indeterminacy** *(NeuORInd) has been imposed.*

§3.5. Gluings, indeterminacies, and pilot discrepancy

As discussed in §3.4, the **Θ-link** involves a **gluing**

$$\{{}^{\dagger}\underline{\underline{q}}_{\underline{v}}^{j^2}\} \;\mapsto\; {}^{\ddagger}\underline{\underline{q}}_{\underline{v}}$$

that identifies ${}^{\ddagger}\underline{\underline{q}}_{\underline{v}}$ [i.e., $2l$-th roots of the *q-parameters* at primes of multiplicative reduction of the [copy belonging to ${}^{\ddagger}\mathfrak{Ring}$ of the] elliptic curve under consideration] with elements, i.e., the ${}^{\dagger}\underline{\underline{q}}_{\underline{v}}^{j^2}$'s, which, when $j \neq 1$, have *different valuations* from the valuation of ${}^{\ddagger}\underline{\underline{q}}_{\underline{v}}$.

On the other hand, in inter-universal Teichmüller theory, by applying the **multiradial representation** of [IUTchIII], Theorem 3.11, which involves various **indeterminacies** (Ind1), (Ind2), (Ind3), and then forming [cf. [IUTchIII], Corollary 3.12, and its proof] the **holomorphic hull** of the union of possible images of the Θ-pilot in this multiradial representation,

(ΘGl) one may treat **both** sides of the **Θ-link gluing** of the above display as belonging to a **single ring theory** without disturbing [cf. the *crucial* **AND relator "∧"** property discussed in §3.4!] the gluing.

Alternative ways to understand the *essential mathematical content* of (ΘGl) include the following:

(NonInf) One may think of (ΘGl) as a statement concerning the **mutually non-interference** or **simultaneous executatibility** of the **Kummer theories** surrounding the ***q*-pilot** and **Θ-pilot** relative to the **gluing** of *abstract* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* constituted by the Θ-link, i.e., when the Kummer theory surrounding the q-pilot is held **fixed**, and one allows the Kummer theory surrounding the Θ-pilot to be subject to various **indeterminacies**.

(Cohab) One may think of (ΘGl) as a statement concerning the **"cohabitation"**, or **"coexistence"**, of the ***q*-pilot** and **Θ-pilot** — relative to the **gluing** of *abstract* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* constituted by the Θ-link — within the **common container** obtained by applying the multiradial representation of the Θ-pilot, forming the **holomorphic hull** [relative to the holomorphic structure [i.e., $(\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-)Hodge theater] that gave rise to the q-pilot under consideration], and finally taking **log-volumes**.

In this context, it is important to recall that this *sort of phenomenon* — i.e.,

> of **computations** of **global degrees/heights** of elliptic curves in situations where a certain **"confusion"**, up to suitable **indeterminacies**, is allowed between *q-parameters* of the elliptic curves and certain *large positive powers* of these q-parameters [i.e., as in (ΘGl)]

— may be seen in various *classical examples* such as

- the proof by Faltings of the **invariance** of **heights** of *abelian varieties* under **isogeny** [cf. the discussion of [Alien], §2.3, §2.4, as well as the discussion of Example 3.2.1 in the present paper],

- the classical proof in characteristic zero of the *geometric version* of the *Szpiro inequality* via the *Kodaira-Spencer morphism*, phrased in terms of the theory of **crystals** [cf. the discussion of [Alien], §3.1, (v)], and
- **Bogomolov's proof** over the complex numbers of the *geometric version* of the *Szpiro inequality* [cf. the discussion of [Alien], §3.10, (vi)]

— cf. also the discussion of [Rpt2018], §16. Moreover, in the case of *crystals*, we observe that, relative to the notation introduced in Example 2.4.5, (v), (vi), we have *correspondences* as follows:

(CrAND) The *logical AND "$\wedge$"* that appears in the *multiradial representation* of the Θ-pilot in **IUT (= AND-IUT)** may be understood as being analogous to the fact that **crystals**, i.e., **"$\wedge$-crystals"**, may be thought of as objects [on infinitesimal neighborhoods of the diagonal inside products of two copies of the scheme under consideration] that may be **simultaneously** interpreted, up to isomorphism, as pull-backs via one projection morphism **and** [cf. "$\wedge$"!] as pull-backs via the other projection morphism [cf. the discussion of ($\wedge(\dot{\vee})$-Chn1) in §3.10 below; the discussion of [Alien], §3.11, (iv), (2^{and}), concerning the interpretation of the discussion of **crystals** in [Alien], §3.1, (v), (3^{KS}), in terms of the logical relator "$\wedge$"].

(CrOR) Thus, from the point of view of the analogy discussed in (CrAND), the *logical OR "$\vee$"* that appears throughout **OR-IUT** may be understood as corresponding to working with **"$\vee$-crystals"** [i.e., as opposed to *crystals (= $\wedge$-crystals)*], that is to say, with objects [on infinitesimal neighborhoods of the diagonal inside products of two copies of the scheme under consideration] that may be interpreted, up to isomorphism, as pull-backs via one projection morphism **or** [cf. "$\vee$"!] as pull-backs via the other projection morphism. Here, we observe that this defining "$\vee$" condition of an $\vee$-crystal is **essentially vacuous** since one may obtain $\vee$-crystals from arbitrary objects on the scheme under consideration simply by pulling back such an object to the infinitesimal neighborhood of the diagonal under consideration via one of the two projection morphisms.

(CrRCS) In a similar vein, from the point of view of the analogies discussed in (CrAND) and (CrOR), **RCS-IUT** may be understood as corresponding to the modified version of the usual theory of crystals obtained by **replacing** the *infinitesimal neighborhoods of the diagonal* inside products of two copies of the scheme under consideration [i.e., that appear in the *usual theory of crystals*!] by the **diagonal itself**. Such a replacement clearly renders the usual theory of crystals **trivial/meaningless**, in a fashion that is essentially very *similar* to the *triviality of $\vee$-crystals* discussed in (CrOR). Finally, we observe that this *similarity* between the modified versions of the usual theory of crystals discussed in (CrOR) and the present (CrRCS) is *entirely analogous* to the *equivalence* **OR-IUT $\Longleftrightarrow$ RCS-IUT** observed in Example 2.4.5, (v), (XOR/RCS).

Unfortunately, however, the situation summarized above in (ΘGl) has resulted in certain frequently voiced **misunderstandings** by some mathematicians. One such *frequently voiced misunderstanding* is to the effect that

(CnfInd1+2) the situation summarized in (ΘGl) may be explained as a consequence of a **"confusion"** between q-parameters and large positive powers of these q-parameters that results from the indeterminacies (Ind1), (Ind2).

In fact, however, as discussed in Example 3.5.1, (iii), below,

> at least in the case of q-parameters of sufficiently small valuation [i.e., sufficiently large positive *order*, in the sense of *loc. cit.*], such a **"confusion"** [i.e., between q-parameters and large positive powers of these q-parameters] can **never occur** as a consequence of (Ind1), (Ind2), i.e., both of which amount to *automorphisms* of the [underlying topological module of the] *log-shells* involved

[cf. also the discussion of (ΘInd) in [Rpt2018], §11]. In this context, we note that this misunderstanding (CnfInd1+2) appears to be caused in many cases, at least in part, by a more *general misunderstanding* concerning the operation of *passage to underlying structures* [cf. Example 3.5.2 below]. A more detailed discussion of the operation of *passage to underlying structures* may be found in §3.9 below.

As discussed in [Rpt2018], §11, the "confusion" summarized in (ΘGl) occurs in inter-universal Teichmüller theory as a consequence not only of the **local indeterminacies** (Ind1), (Ind2), (Ind3), but also of the constraints imposed by the **global realified Frobenioid** portions of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the Θ-link. In this context, it is of *particular importance* to observe that

(CnfInd3) the indeterminacy **(Ind3)**, which constrains one to restrict one's attention to **upper bounds** [i.e., but **not** lower bounds!] on the log-volume that is the subject of the computation of [IUTchIII], Corollary 3.12, already by itself — i.e., *without considering (Ind1), (Ind2), or global realified Frobenioids!* [cf. the discussion of (Ind3>1+2) in §3.11 below] — is **sufficient to account** for the possibility of a **"confusion"** of the sort summarized in (ΘGl) [i.e., between q-parameters and large positive powers of these q-parameters].

Indeed, the indeterminacy (Ind3) is *defined* in precisely such a way as to **identify** the ideals generated by *arbitrary positive powers* of the q-parameters.

Example 3.5.1: Bounded nature of log-shell automorphism indeterminacies. Write $\mathbb{Z}_p$ for the ring of p-adic integers, for some prime number p; $\mathbb{Q}_p$ for the field of fractions of $\mathbb{Z}_p$.

(i) Let M be a finitely generated free $\mathbb{Z}_p$-module, which, in the following discussion, we shall think of as being embedded in $M_{\mathbb{Q}_p} \stackrel{\text{def}}{=} M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$;

$$\alpha : \ M \ \stackrel{\sim}{\to} \ M$$

an *automorphism* of the $\mathbb{Z}_p$-module M. For $n \in \mathbb{Z}$, write

$$\mathcal{U}(M, n) \ \stackrel{\text{def}}{=} \ \{x \in M_{\mathbb{Q}_p} \mid x \in p^n \cdot M, \ x \notin p^{n+1} \cdot M\} \ \subseteq \ M_{\mathbb{Q}_p}.$$

Then observe that α induces a *bijection*

$$\mathcal{U}(M, n) \ \stackrel{\sim}{\to} \ \mathcal{U}(M, n)$$

for every $n \in \mathbb{Z}$.

(ii) In the notation of (i), suppose, for simplicity, that p is *odd*. Let k be a finite field extension of $\mathbb{Q}_p$. Write $\mathcal{O}_k \subseteq k$ for the ring of integers of k; $\mathcal{O}_k^\times \subseteq \mathcal{O}_k$ for the group of units of $\mathcal{O}_k$; $\mathfrak{m}_k \subseteq \mathcal{O}_k$ for the maximal ideal of k; $\mathcal{I}_k \subseteq k$ for the *log-shell* associated to k [cf., e.g., the discussion of [IUTchIII], Remark 1.2.2, (i)], i.e., the result of multiplying by p^{-1} the image $\log_p(\mathcal{O}_k^\times)$ of $\mathcal{O}_k^\times$ by the p-adic logarithm $\log_p(-)$. Thus,

$$\mathcal{O}_k \ \subseteq \ \mathcal{I}_k \ \subseteq \ p^{-c} \cdot \mathcal{O}_k$$

for some nonnegative integer of c that depends only on the isomorphism class of the field k [cf. [IUTchIV], Proposition 1.2, (i)]. In particular, there exists a positive integer s that depends only on the isomorphism class of the field k such that for any *automorphism*

$$\phi : \ \mathcal{I}_k \ \stackrel{\sim}{\to} \ \mathcal{I}_k$$

of the $\mathbb{Z}_p$-module $\mathcal{I}_k$ and any $n \in \mathbb{Z}$, it holds that

$$\phi(\mathcal{U}(\mathcal{O}_k, n)) \ \subseteq \ \bigcup_{i=-s}^{s} \mathcal{U}(\mathcal{O}_k, n+i)$$

[where i ranges over the integers between $-s$ and s].

(iii) In the situation of (ii), we define the *order* of a nonzero element $x \in k$ to be the unique $n \in \mathbb{Z}$ such that $x \in \mathfrak{m}_k^n$, $x \notin \mathfrak{m}_k^{n+1}$. One thus concludes from the final portion of the discussion of (ii) that there exists a positive integer t that depends only on the isomorphism class of the field k such that for any *automorphism*

$$\phi : \ \mathcal{I}_k \ \stackrel{\sim}{\to} \ \mathcal{I}_k$$

of the $\mathbb{Z}_p$-module $\mathcal{I}_k$ and any nonzero element $q \in \mathcal{O}_k$ [i.e., such as the *q-parameter* of a Tate curve over k!], the absolute value of the *difference* between the *orders* of q and $\phi(q)$ is $\leq t$, i.e., in words,

> automorphisms of the $\mathbb{Z}_p$-module $\mathcal{I}_k$ only give rise to **bounded discrepancies** in the orders of nonzero elements of $\mathcal{O}_k$.

Example 3.5.2: Examples of gluings. *Distinct auxiliary structures* on some **common** [i.e., "$\wedge$"!] *underlying structure* may be thought of as **gluings** of the distinct auxiliary structures along the common underlying structure. Here, we observe that, in general, *distinct auxiliary structures* on a *common underlying structure* are *not* necessarily mapped to one another by some *automorphism of the common underlying structure.* Concrete examples of these generalities may be found in quite substantial abundance throughout arithmetic geometry and include, in particular, the examples (i), (ii), (iii), (iv), (v) given below, as well as the elementary Examples 2.3.2, 2.4.1, 2.4.2, 2.4.3, 2.4.7, 2.4.8, 3.3.1 discussed in §2.3, §2.4, §3.3 [cf. also the discussion of [Rpt2018], §11]. In passing, we observe that

> these examples may also be understood as interesting examples of the sort of **gluing/logical AND "$\wedge$" relation** that appears in the **Θ-/$\mathfrak{log}$-links** of inter-universal Teichmüller theory, i.e., examples of situations that

are *qualitatively similar* to the Θ-/$\mathfrak{log}$-links of inter-universal Teichmüller theory in the sense that they involve *distinct auxiliary structures* that are *glued together* along some *common auxiliary structure*

[cf. the discussion of (StR1) $\sim$ (StR6) in §3.2; the discussion of §3.4; the portion of the present §3.5 preceding Example 3.5.1].

(i) The group structures of the **finite abelian groups** $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ are *not* mapped to one another by any *isomorphism of sets*, despite the fact that the underlying sets of these two groups are indeed isomorphic to one another. This example is also of interest in light of the discussion of **truncated Witt vectors** in Example 2.4.6, (iii).

(ii) The *scheme structures* of **non-isomorphic algebraic curves** over a common algebraically closed field are *not* mapped to one another by any *isomorphism of topological spaces*, despite the fact that the underlying topological spaces of algebraic curves over a common algebraically closed field are indeed isomorphic to one another.

(iii) The *holomorphic structures* of **non-isomorphic compact Riemann surfaces** R_1, R_2 with *homeomorphic* underlying topological spaces are *not* mapped to one another by any *homeomorphism*, i.e., by any *isomorphism of topological spaces*, despite the fact that the underlying topological spaces of such Riemann surfaces R_1, R_2 are indeed isomorphic to one another. [This example is in fact *essentially similar* to the situation discussed in Example 3.3.1, except that in the situation of Example 3.3.1, the *"two" Riemann surfaces* involved are both isomorphic to the *complex plane*, hence, in particular, *isomorphic* to one another.] In this context, it is of interest to observe that

(iii-a) if one defines the **genus** of such a compact Riemann surface R_i, for $i \in \{1, 2\}$, as the complex dimension of the space of *global holomorphic differentials* on the Riemann surface, then it is *by no means clear* that R_1 and R_2 have the *same genus*, i.e., since this definition of the genus depends, in an essential way, on the **holomorphic structure** of the Riemann surface.

On the other hand, once one verifies that

(iii-b) this "holomorphic definition" of the genus coincides with the genus defined in terms of the singular homology group of the **underlying topological space**, it follows immediately that R_1 and R_2 do indeed have the *same genus*.

This contrast between the "holomorphic" and "topological" definitions of the genus is of interest in the context of inter-universal Teichmüller theory since it illustrates

(iii-c) the *very substantive significance* of formulating the definitions of objects or constructions [i.e., in the present discussion, the "genus"] in terms of structures that are **coric** for the **"gluing/link"** [i.e., in the present discussion, a comparison of distinct holomorphic structures on homeomorphic topological spaces] under consideration, that is to say, in terms of structures that are **commonly shared** in an **invariant** fashion by, hence satisfy a **logical AND "$\wedge$" relation** relative to, the two objects [i.e., in the present discussion, R_1 and R_2] that are related to one another by the link under consideration.

We refer to the discussion of §3.8 below for a more detailed treatment of the importance of coric structures in inter-universal Teichmüller theory.

(iv) The *field structures* of **non-isomorphic mixed-characteristic local fields** [which, by *local class field theory*, may be regarded as [the formal union with "$\{0\}$" of] some suitable subquotient of their respective absolute Galois groups] are *not*, in general, mapped to one another by any *isomorphism of profinite groups* between the respective absolute Galois groups [cf., e.g., [Ymgt], §2, Theorem, for an example of this phenomenon].

(v) In the notation of Example 3.5.1, (i), let X be a *proper smooth curve of genus* ≥ 2 over $\mathbb{F}_p \stackrel{\mathrm{def}}{=} \mathbb{Z}_p/p\mathbb{Z}_p$. Thus, X may be thought of as an *"underlying structure"* associated to any *lifting of* X *to* $\mathbb{Z}_p$, i.e., any flat $\mathbb{Z}_p$-scheme Y equipped an isomorphism of $\mathbb{F}_p$-schemes $Y \times_{\mathbb{Z}_p} \mathbb{F}_p \stackrel{\sim}{\to} X$. Then observe that **non-isomorphic liftings** of X to $\mathbb{Z}_p$ are *not*, in general, mapped to one another by any *automorphism of the* $\mathbb{F}_p$*-scheme* X. [Indeed, this is particularly easy to see if one chooses X such that X does not admit any nontrivial automorphisms.] In passing, we note that this example may be regarded as a sort of *p-adic analogue* of the example of (iii). Finally, we make the important *observation* that **crystals** on X [relative to $\mathbb{Z}_p$] are objects that are **coric/common** to *arbitrary liftings* of X to $\mathbb{Z}_p$. This *observation* is of particular importance in light of the *strong structural resemblances* between inter-universal Teichmüller theory and the *theory of crystals* [cf. [Alien], §3.1, (v); the discussion of (CrAND) in the present §3.5; the discussion of §3.10 below].

Example 3.5.3: Gluings from the point of view of tilts of perfectoid fields. Certain aspects of

(UGΘ) the **unit group portions** of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the **Θ-link** of inter-universal Teichmüller theory

bear a certain resemblance to

(TltPh) certain phenomena that occur in the theory of **tilts** of **perfectoid fields** [cf., e.g., [Bns], §5, for an exposition of basic facts surrounding this theory].

In the present Example 3.5.3, we discuss *similarities* and *differences* between (UGΘ) and (TltPh).

(i) Fix a prime number p. Let k be a *perfectoid field* of *characteristic zero* [cf. [Bns], Definition 5.1.1], $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$; $\mathbb{C}_k$ for the completion of $\overline{k}$, which is itself a perfectoid field [cf. [Bns], Example 5.1.2, (2)]; "$\mathcal{O}_{(-)}$" for the ring of integers of a field equipped with an absolute value; $\mathbb{Z}_p(1) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mathcal{O}^{\times}_{\overline{k}})$; $\mathbb{Q}_p(1) \stackrel{\mathrm{def}}{=} \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then we recall from [Bns], §5.2.2; [Bns], Proposition 5.2.3 [cf. also [Bns], Example 5.1.2, (1)], that the *tilt* $k^\flat$ of k is a *perfectoid field* of *characteristic* p whose ring of integers $\mathcal{O}_{k^\flat}$ admits *natural multiplicative bijections* [i.e., bijections that are compatible with the respective multiplicative structures]

$$\mathcal{O}_{k^\flat} \quad \stackrel{\sim}{\to} \quad \varprojlim \ \mathcal{O}_k \quad \stackrel{\sim}{\to} \quad \varprojlim \ \mathcal{O}_k/p \cdot \mathcal{O}_k$$

— where the inverse systems implicit in the inverse limit are indexed by $\mathbb{N}$ with transition morphisms given by the *p-th power map* [cf. also the discussion of (v) below], and the second "$\overset{\sim}{\to}$" is given by reduction modulo p. Write $\overline{k}^{\flat}$ for the direct limit of the finite extensions of $k^{\flat}$ determined by the tilts of the finite subextensions of the extension $\overline{k}/k$ [cf. [Bns], Theorem 5.4.4]. In particular, we conclude, by replacing "$\mathcal{O}$" by "$\mathcal{O}^{\times}$" and then considering the quotients "$\mathcal{O}^{\times\boldsymbol{\mu}}$" [i.e., the units modulo torsion] and "$\mathcal{O}^{\times\boldsymbol{\mu}'}$" [i.e., the units modulo prime-to-p torsion], that we have *natural isomorphisms* and a *natural exact sequence* of *[multiplicative] topological modules*

$$\mathcal{O}^{\times}_{k^{\flat}} \quad \overset{\sim}{\to} \quad \varprojlim \ \mathcal{O}^{\times}_{k}; \qquad \mathcal{O}^{\times\boldsymbol{\mu}}_{\mathbb{C}^{\flat}_{k}} \ = \ \mathcal{O}^{\times\boldsymbol{\mu}'}_{\mathbb{C}^{\flat}_{k}} \quad \overset{\sim}{\to} \quad \mathcal{O}^{\times\widetilde{\boldsymbol{\mu}}}_{\mathbb{C}_{k}} \ \overset{\mathrm{def}}{=} \ \varprojlim \ \mathcal{O}^{\times\boldsymbol{\mu}'}_{\mathbb{C}_{k}};$$

$$1 \ \longrightarrow \ \mathbb{Q}_p(1) \ \longrightarrow \ \mathcal{O}^{\times\widetilde{\boldsymbol{\mu}}}_{\mathbb{C}_{k}} \ \longrightarrow \ \mathcal{O}^{\times\boldsymbol{\mu}}_{\mathbb{C}_{k}} \ \longrightarrow \ 1,$$

all of which are compatible with the respective natural actions of $G_{k^{\flat}} \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}^{\flat}/k^{\flat})$ and G_k, relative to the *natural isomorphism* $G_{k^{\flat}} \overset{\sim}{\to} G_k$ induced by the tilt construction [cf. [Bns], Theorem 5.4.4]. Finally, in this context, we observe that we have an *exact sequence of [multiplicative] topological* G_k*-modules*

$$1 \ \longrightarrow \ \mathbb{Q}_p(1) \ \longrightarrow \ \mathcal{O}^{\times\widetilde{\boldsymbol{\mu}}}_{\overline{k}} \ \longrightarrow \ \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \ \longrightarrow \ 1,$$

where we write $\mathcal{O}^{\times\widetilde{\boldsymbol{\mu}}}_{\overline{k}} \overset{\mathrm{def}}{=} \varprojlim \ \mathcal{O}^{\times\boldsymbol{\mu}'}_{\overline{k}}$. Thus, we have a natural G_k*-equivariant injection* $\mathcal{O}^{\times\widetilde{\boldsymbol{\mu}}}_{\overline{k}} \hookrightarrow \mathcal{O}^{\times\widetilde{\boldsymbol{\mu}}}_{\mathbb{C}_{k}}$ that has *dense image*, but is *not surjective*.

(ii) The discussion of (i) may be summarized as follows. In the notation of (i), we obtain a *natural isomorphism of [multiplicative] topological modules* equipped with continuous actions by *topological groups*

$$\begin{tikzcd}
G_{k^{\flat}} \arrow[r, "\sim"] & G_k \\
\curvearrowright & \curvearrowright \\
\mathcal{O}^{\times\boldsymbol{\mu}}_{\mathbb{C}^{\flat}_{k}} \arrow[r, "\sim"] & \mathcal{O}^{\times\widetilde{\boldsymbol{\mu}}}_{\mathbb{C}_{k}}.
\end{tikzcd}$$

In particular, if ${}^{\dagger}k$ and ${}^{\ddagger}k$ are perfectoid fields of characteristic zero, and ${}^{\dagger}k^{\flat} \overset{\sim}{\to} {}^{\ddagger}k^{\flat}$ is an isomorphism of topological fields, then, by composing the diagrams as above in the case of ${}^{\dagger}k$ and ${}^{\ddagger}k$, we obtain a *natural isomorphism of [multiplicative] topological modules* equipped with continuous actions by *topological groups*

$$\begin{tikzcd}
G_{{}^{\dagger}k} \arrow[r, "\sim"] & G_{{}^{\ddagger}k} \\
\curvearrowright & \curvearrowright \\
\mathcal{O}^{\times\widetilde{\boldsymbol{\mu}}}_{\mathbb{C}_{{}^{\dagger}k}} \arrow[r, "\sim"] & \mathcal{O}^{\times\widetilde{\boldsymbol{\mu}}}_{\mathbb{C}_{{}^{\ddagger}k}}.
\end{tikzcd}$$

It is well-known that there are many examples of such ${}^{\dagger}k$, ${}^{\ddagger}k$, such as suitable pairs of the following perfectoid fields [cf. [Bns], Example 4.1.5, (1), (2); [Bns], Theorem 5.1.4] associated to a finite unramified extension E of $\mathbb{Q}_p$:

(TltEx1) the p-adic completion of $E(\zeta_{p^\infty})$, i.e., the field extension of E obtained by adjoining all p-power roots of unity;

(TltEx2) the p-adic completion of $E(\pi^{\frac{1}{p^\infty}})$, i.e., the field extension of E obtained by adjoining to E a system of p-power roots of a uniformizer π of E.

Finally, we observe that in the situation where k is the p-adic completion of some [necessarily infinite!] subextension of some algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, we obtain *compatible natural inclusions* $G_k \hookrightarrow G_{\mathbb{Q}_p} \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $\mathcal{O}^{\widetilde{\times\boldsymbol{\mu}}}_{\overline{\mathbb{Q}}_p} \hookrightarrow \mathcal{O}^{\widetilde{\times\boldsymbol{\mu}}}_{\mathbb{C}_k}$. Here, the image of the first inclusion $G_k \hookrightarrow G_{\mathbb{Q}_p} \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is *closed*, but *not open* [i.e., is necessarily of infinite index], while the second inclusion $\mathcal{O}^{\widetilde{\times\boldsymbol{\mu}}}_{\overline{\mathbb{Q}}_p} \hookrightarrow \mathcal{O}^{\widetilde{\times\boldsymbol{\mu}}}_{\mathbb{C}_k}$ has *dense image*, but is *not surjective*.

(iii) Next, let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$; $K \subseteq K_1 \subseteq \ldots \subseteq K_i \subseteq K_{i+1} \subseteq \ldots \subseteq \overline{\mathbb{Q}}_p$ a sequence, indexed by the positive integers, of finite extensions of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ such that

(iii-a) there exists a positive integer i_0 such that for all $i \geq i_0$, K_{i+1} is a *totally ramified* extension of K_i of *degree* p;

(iii-b) $K_\infty \overset{\mathrm{def}}{=} \cup_{i\geq 1} K_i$ is *deeply ramified* [cf. [Bns], Definition 4.1.4].

For each positive integer i, write $G_{K_i} \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/K_i)$; $G_{K_\infty} \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/K_\infty)$; $\underline{K}_i$, $\underline{K}_\infty$, $\overline{\mathbb{F}}_p$ for the respective residue fields of K_i, K_∞, $\overline{\mathbb{Q}}_p$. One verifies immediately that if $L \subseteq \overline{\mathbb{Q}}_p$ is any finite extension of $\mathbb{Q}_p$, then the sequence of composite extensions $L \subseteq L \cdot K_1 \subseteq \ldots \subseteq L \cdot K_i \subseteq L \cdot K_{i+1} \subseteq \ldots \subseteq \overline{\mathbb{Q}}_p$ satisfies the same conditions as the $\{K_i\}_{i\geq 1}$. Observe that by *local class field theory*, we have *natural isomorphisms*

$$G^{\mathrm{ab}}_{K_\infty} \overset{\sim}{\to} \varprojlim_i G^{\mathrm{ab}}_{K_i} \overset{\sim}{\to} \varprojlim_i (K_i^\times)^\wedge$$

— where the superscript "ab" denotes the topological abelianization of a topological group; "$\wedge$" denotes the profinite completion; the indices "i" in the inverse limits range over the positive integers; the transition morphisms in the first inverse limit are the natural morphisms induced by the *natural inclusions* $G_{K_{i+1}} \hookrightarrow G_{K_i}$; the transition morphisms in the second inverse limit are the morphisms induced by the *norms* of the extensions K_{i+1}/K_i. Thus, it follows immediately from our assumption (iii-a) on the $\{K_i\}_{i\geq 1}$, together with *local class field theory*, that the residue field $\underline{K}_\infty$ of K_∞ is *finite*, and that we have *natural isomorphisms*

$$\underline{K}^\times_\infty \overset{\sim}{\to} \boldsymbol{\mu}_{p'}(G^{\mathrm{ab}}_{K_\infty}); \quad \overline{\mathbb{F}}^\times_p \overset{\sim}{\to} \varinjlim_H \boldsymbol{\mu}_{p'}(H^{\mathrm{ab}})$$

— where the superscript "$\times$" denotes the group of nonzero elements of a field; "$\boldsymbol{\mu}_{p'}(-)$" denotes the subgroup of prime-to-p torsion elements of an abelian group; the "H" in the direct limit ranges over the open subgroups of G_{K_∞}. In particular, the direct limit of the above display yields a *functorial group-theoretic algorithm*, whose input data is the topological group G_{K_∞}, and whose functoriality is with respect to isomorphisms of topological groups, for *reconstructing the* G_{K_∞}*-module* $\overline{\mathbb{F}}^\times_p$. Since the kernel of the action of G_{K_∞} on $\overline{\mathbb{F}}^\times_p$ may be identified with the *inertia subgroup* $I_{K_\infty} \subseteq G_{K_\infty}$, we thus obtain

(RcnCh) a *functorial group-theoretic algorithm* in the topological group G_{K_∞} for *reconstructing the set of totally ramified characters with open image* $\chi : G_{K_\infty} \to \mathbb{Z}_p^\times$.

Next, observe that [cf. [Bns], Theorem 5.1.4; our assumption (iii-b) on the $\{K_i\}_{i\geq 1}$] we make take the *perfectoid field* k of (i) to be the *p-adic completion* of K_∞ and identify G_{K_∞} with G_k [cf. [Bns], Theorem 1.1.8]. In particular, we recall [cf., e.g., [Bns], Corollary 16.1.3; [Bns], Example 16.1.4] that the *p-adic logarithm* determines a *G_k-equivariant isomorphism of topological modules* $\mathcal{O}^{\times\boldsymbol{\mu}}_{\mathbb{C}_k} \xrightarrow{\sim} \mathbb{C}_k$. Now let us consider the *perfectoid fields* ${}^\dagger k$, ${}^\ddagger k$ of (TltEx1), (TltEx2) [i.e., associated to some finite unramified extension E of $\mathbb{Q}_p$ — cf. (ii)], which, as is easily verified, may be constructed in the fashion of the perfectoid field "k" of the present discussion. Write ${}^\dagger\chi_{\text{cyc}} : G_{{}^\dagger k} \to \mathbb{Z}_p^\times$, ${}^\ddagger\chi_{\text{cyc}} : G_{{}^\ddagger k} \to \mathbb{Z}_p^\times$ for the respective *p-adic cyclotomic characters* of ${}^\dagger k$, ${}^\ddagger k$. Thus, ${}^\dagger\chi_{\text{cyc}}$ is, by definition [cf. (TltEx1)], *trivial*, while ${}^\ddagger\chi_{\text{cyc}}$, by definition [cf. (TltEx2)], has *open image*. Moreover, the *manifestly intrinsically distinct nature* of ${}^\dagger\chi_{\text{cyc}}$, ${}^\ddagger\chi_{\text{cyc}}$ implies that

(DstCh) any isomorphism $\mathcal{O}^{\widetilde{\times\boldsymbol{\mu}}}_{\mathbb{C}_{{}^\dagger k}} \xrightarrow{\sim} \mathcal{O}^{\widetilde{\times\boldsymbol{\mu}}}_{\mathbb{C}_{{}^\ddagger k}}$ arising from a diagram of isomorphisms as in the *second display* of (ii) *fails to induce* an isomorphism of the respective *exact sequences*

$$1 \longrightarrow \mathbb{Q}_p(1) \longrightarrow \mathcal{O}^{\widetilde{\times\boldsymbol{\mu}}}_{\mathbb{C}_{{}^\dagger k}} \longrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}_{\mathbb{C}_{{}^\dagger k}} \longrightarrow 1,$$

$$1 \longrightarrow \mathbb{Q}_p(1) \longrightarrow \mathcal{O}^{\widetilde{\times\boldsymbol{\mu}}}_{\mathbb{C}_{{}^\ddagger k}} \longrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}_{\mathbb{C}_{{}^\ddagger k}} \longrightarrow 1,$$

of (i) associated to ${}^\dagger k$, ${}^\ddagger k$; in particular, by applying the p-adic logarithm $\mathcal{O}^{\times\boldsymbol{\mu}}_{\mathbb{C}_{{}^\dagger k}} \xrightarrow{\sim} \mathbb{C}_{{}^\dagger k}$, we conclude that the *totally ramified character with open image* $({}^\dagger\chi_{\text{cyc}} \neq)\ \psi \overset{\text{def}}{=} {}^\ddagger\chi_{\text{cyc}}|_{G_{{}^\dagger k}} : G_{{}^\dagger k} \to \mathbb{Z}_p^\times$ [cf. (RcnCh)] satisfies the *remarkable nonvanishing property*

$$H^0(G_{{}^\dagger k}, \mathbb{C}_{{}^\dagger k}(\psi^{-1})) \neq 0$$

— i.e., in sharp contrast to the vanishing of the corresponding cohomology module in the well-known classical situation where "${}^\dagger k$" is replaced by a *finite extension* of $\mathbb{Q}_p$ [cf., e.g., [Bns], Theorem 4.3.2, (iii)].

Put another way, relative to a diagram of isomorphisms as in the *second display* of (ii),

(DstHT) the perfectoid fields ${}^\dagger k$, ${}^\ddagger k$ do *not* share a *commmon p-adic cyclotomic character* [i.e., in sharp contrast to the situation for finite extensions of $\mathbb{Q}_p$ — cf. [AbsAnab], Proposition 1.2.1, (vi)] or *common properties* involving *vanishing of cohomology groups* — all of which constitute fundamental aspects of *p-adic Hodge theory*; that is to say, in a word, the perfectoid fields ${}^\dagger k$, ${}^\ddagger k$ do **not** share a **"common *p*-adic Hodge theory"**.

In particular, any aspects of anabelian geometry that involve conventional techniques of p-adic Hodge theory cannot be applied to isomorphisms as in the *second display* of (ii).

(iv) Next, we observe, in the context of the discussion of the final portion of (ii), that

(TltSim) the diagrams in the two displays of (ii) are *substantially reminiscent* of the isomorphisms between the *unit group portions* [cf., e.g., the discussion of "(a^{Θ})" and "(a^{q})" in [Alien], §3.3, (vii)] of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the **Θ-link** of inter-universal Teichmüller theory.

Of course, as observed in (ii) and (iii), there are *immediately evident differences*, such as the fact that

(TltDf1) unlike the situation with the unit group portion of the Θ-link of inter-universal Teichmüller theory, the image of the inclusion $G_k \hookrightarrow G_{\mathbb{Q}_p} \overset{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is *closed*, but *not open*, while the inclusion $\mathcal{O}^{\times\widetilde{\boldsymbol{\mu}}}_{\overline{\mathbb{Q}}_p} \hookrightarrow \mathcal{O}^{\times\widetilde{\boldsymbol{\mu}}}_{\mathbb{C}_k}$ involves "$\mathcal{O}^{\times\widetilde{\boldsymbol{\mu}}}$" instead of "$\mathcal{O}^{\times\boldsymbol{\mu}}$" [cf. the discussion of (DstCh), (DstHT) in the final portion of (iii)!] and has *dense image*, but is *not surjective*.

Another *fundamental difference*, in the case of the first display of (ii), lies in the difference between notions of *globality*:

(TltDf2) the fact that the *tilt* $k^\flat$ is of *positive characteristic* means that,

- unlike the case with k, which, in the context of inter-universal Teichmüller theory [cf. the discussion of the final portion of (ii)], is regarded as some sort of *localization* of [i.e., more precisely, the p-adic completion of an infinite extension of] a **number field**,
- the only natural way to regard $k^\flat$ as some sort of localization of a *global field* is to regard it as a *localization* of [i.e., more precisely, a completion of an infinite extension of] a **one-dimensional function field** over a base field of **positive characteristic**.

That is to say, unlike the case with inter-universal Teichmüller theory, the sort of "link" constituted by the first display of (ii) is necessarily a "link" between local data arising from *two fundamentally different notions of globality.*

(v) Before proceeding further, it is interesting to note, in the context of the *general themes* of **"redundant copies"** and **gluings** [cf. the discussion of §2.3, §2.4, §3.1; Example 3.5.2; the present Example 3.5.3], that each of the *transition morphisms* "$\mathcal{O}_k \to \mathcal{O}_k$" in the inverse limit

$$\mathcal{O}_{k^\flat} \quad = \quad \varprojlim \ \mathcal{O}_k$$

used to define the *tilt* [cf. (i)] may be regarded as a *"gluing"* between a certain quotient of the *copy* of $\mathcal{O}_k$ in the *domain* and a certain subset of the *copy* of $\mathcal{O}_k$ in the *codomain.* Note, moreover, that although

(RCTlt1) these domain and codomain copies of $\mathcal{O}_k$ admit a **ring isomorphism**, i.e., if one does not require any sort of *compatibility* of the isomorphism, in the evident sense, with the transition morphism [i.e., raising to the p-th power],

(RCTlt2) these domain and codomain copies of $\mathcal{O}_k$ do **not**, in general, admit an

isomorphism [as rings or indeed even as sets!] that is *compatible*, in the evident sense, with the transition morphism [i.e., raising to the p-th power].

Moreover, if one takes the **"RCS point of view"** of asserting that the domain and codomain copies of $\mathcal{O}_k$ are **"redundant"**, hence may be **identified**, then the resulting inverse limit is simply the set

$$\{x \in \mathcal{O}_k \mid x^p = x\}$$

— i.e., a set that is *manifestly completely different* from the set obtained by taking the inverse limit as in the definition of the tilt in (i), i.e., where one *distinguishes the implicit distinct labels/indices* of copies of $\mathcal{O}_k$ that appear in the inverse system.

(vi) In the context of the *general themes* of **"redundant copies"** and **gluings** [cf. the discussion of §2.3, §2.4, §3.1; Example 3.5.2; the present Example 3.5.3], we observe that (RCTlt1), (RCTlt2) have the following respective *precise analogues* in the theory of *holomorphic structures* on *Riemann surfaces* [cf. the discussion of Examples 3.3.1; 3.5.2, (iii)]:

(RCRS1) whereas the domain and codomain copies of $\mathbb{C}$ in the *fundamental Teichmüller deformation* Λ of Example 3.3.1 admit a **holomorphic isomorphism**, i.e., if one does not require any sort of *compatibility* of the isomorphism, in the evident sense, with the homeomorphism Λ,

(RCRS2) these domain and codomain copies of $\mathbb{C}$ do **not** admit a **holomorphic isomorphism** that is *compatible*, in the evident sense, with the homeomorphism Λ.

By contrast, the situation considered in Example 3.5.2, (iii), gives an example of situation in which

(RCRS3) one has *distinct holomorphic structures* on the same underlying compact topological surface that give rise to Riemann surfaces R_1, R_2 that do **not** admit a **holomorphic isomorphism**, i.e., *regardless of whether or not one imposes a compatibility condition* [cf. (RCRS1), (RCRS2)] with some homeomorphism between the underlying topological surfaces of R_1, R_2.

Note, moreover, that (RCRS1), (RCRS2) [or, alternatively, (RCTlt1), (RCTlt2)] admit the following respective *precise analogues* in the theory surrounding the **Θ-link** in inter-universal Teichmüller theory:

(RCΘ1) whereas the ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters in the domain and codomain of the Θ-*link* in inter-universal Teichmüller theory are **isomorphic**, i.e., if one does not require any sort of *compatibility* of the isomorphism, in the evident sense, with the Θ-link,

(RCΘ2) as observed in Example 3.2.2, (i-a), (iv) [cf. also the discussion of Example 2.4.8, (ii)] — essentially as a consequence of the *definition of a ring*, i.e., the elementary observation that the N*-th power map*, for an integer $N \geq 2$, on a domain of generic characteristic zero is *not a ring homomorphism* (!) — these domain and codomain ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters do **not** admit an **isomorphism** that is *compatible*, in the evident sense, with the Θ-link.

Thus, from the point of view of the analogy discussed in (TltSim),

(RCΘ3) the diagrams in the two displays of (ii) yield a situation analogous to (RCRS3) in the sense that **neither** the pairs

$$G_{k^\flat} \curvearrowright \mathbb{C}_k^\flat; \quad G_k \curvearrowright \mathbb{C}_k$$

[i.e., consisting of a topological field equipped with a continuous action by a topological group] **nor** the pairs

$$G_{{}^\dagger k} \curvearrowright \mathbb{C}_{{}^\dagger k}; \quad G_{{}^\ddagger k} \curvearrowright \mathbb{C}_{{}^\ddagger k}$$

are **isomorphic**, i.e., *regardless of whether or not one imposes a compatibility condition* [cf. (RCΘ1), (RCΘ2)] with the [analogue of the] Θ-link; in particular, this situation yields *interesting counterexamples* to the analogue [i.e., for perfectoid fields of characteristic zero and their tilts] of the *Neukirch-Uchida Theorem for number fields* [cf. the discussion of Example 3.5.2, (iv)].

Here, we note in passing that in fact,

(RCΘ2lg) the property discussed in (RCΘ2) [which is in fact an immediate consequence of the *definition of a ring*!] is **never logically applied** in the development or proofs of the main results [such as, for instance, [IUTchIII], Theorem 3.11, or [IUTchIII], Corollary 3.12] of inter-universal Teichmüller theory.

That is to say, even if one takes the position that one does not know whether or not the property discussed in (RCΘ2) holds, there is *no effect whatsoever* on the *essential logical structure* of the development or proofs of the main results [such as, for instance, [IUTchIII], Theorem 3.11, or [IUTchIII], Corollary 3.12] of inter-universal Teichmüller theory. Rather, the only effect of taking such a position is that it implies that there is a possibility that the theory involves a sort of *"overkill"*, i.e., that one is possibly doing more than is in fact necessary in order to prove the desired results.

(vii) Finally, we observe that

(StrDf) although the observation of (RCΘ3) is of **independent interest** in its own right — e.g., from the point of view of the analogy between inter-universal Teichmüller theory and the theory of **Witt vectors** and ***p*-adic Teichmüller theory** discussed in the final portion of [Alien], §3.3, (ii) [cf. also the discussion of *epiperfect schemes* in [*p*Tch], Chapter VI] — relative to the analogy discussed in (TltSim), the sorts of pairs considered in (RCΘ3) are **completely useless** from the point of view of constructing any sort of theory using these pairs that is in some sense *structurally analogous to inter-universal Teichmüller theory.*

One important aspect of (StrDf) may be seen in the differences discussed in (TltDf1) above [cf. also the discussion surrounding (DstCh), (DstHT) in the final portion of (iii)]. In the context of (TltDf1), it is also of interest to note that, unlike the case of absolute Galois groups of finite extensions of $\mathbb{Q}_p$, which are of *cohomological* *dimension* 2, the fields ${}^{\dagger}k$, ${}^{\ddagger}k$ of (iii) are of *cohomological dimension* 1. Another important aspect of (StrDf) may be seen in the *two fundamentally different notions of globality* discussed in (TltDf2) above. Other important **structural differences**, i.e., aspects of (StrDf), all of which play a **central role** in the *essential logical structure of inter-universal Teichmüller theory*, include the following:

(TltDf3) The **positive characteristic** nature of $k^{\flat}$ means that $k^{\flat}$ does **not** admit any sort of evident analogue of the *p-adic logarithm*, which is *fundamental* to the definition of the **log-link** in inter-universal Teichmüller theory.

(TltDf4) The **positive characteristic** nature of $k^{\flat}$ means that $k^{\flat}$ does **not** admit any sort of evident analogue of the **pro-*p* portion** of the theory of **cyclotomic rigidity** surrounding the **theta function**, as well as local and global fields [cf., e.g., the discussion of [Alien], §3.4, (v)], all of which plays a *fundamental role* in inter-universal Teichmüller theory.

(TltDf5) Any situation such as the one described in (RCΘ3) — i.e., where the *topological fields* in the domain and codomain of the [analogue of the] Θ-link are *not isomorphic* — necessarily **fails** to satisfy, i.e., even at the level of *étale-like objects*, the **symmetry** property discussed in Example 3.2.2, (ii-c), (iv), which plays a *fundamental role* in inter-universal Teichmüller theory, namely, in establishing the **multiradiality** properties that underlie the *multiradial representation* of the Θ-pilot given in [IUTchIII], Theorem 3.11 [cf. the discussion of Example 3.3.1 in (vi); the discussion of the closely related *bijection* $\mathbb{C}^{\times}\backslash GL_2^{+}(\mathbb{R})/\mathbb{C}^{\times} \xrightarrow{\sim} [0,1)$ in (InfH); [Alien], §3.1, (iii); [Alien], §3.2; [Alien], §3.6, (i); [Alien], §3.7, (i)].

(TltDf6) The **non-locally compact** nature of perfectoid fields [i.e., the *non-openness* discuss in (TltDf1)] means that fields such as k, ${}^{\dagger}k$, ${}^{\ddagger}k$ do **not** admit any sort of evident analogue of the phenomenon of **compatibility** of **cyclotomic rigidity isomorphisms** with the **profinite/tempered topology**, which plays a *fundamental role* in inter-universal Teichmüller theory [cf. the discussion of [Alien], §3.4, (v), as well as the discussion in the present paper of *"truncatibility"* in Examples 3.8.3, 3.8.4, below].

(TltDf7) The **non-locally compact** nature of perfectoid fields [i.e., the *non-openness* discuss in (TltDf1)] means that fields such as k, ${}^{\dagger}k$, ${}^{\ddagger}k$ do **not** admit any sort of evident analogue of the notion of **log-volume**, which plays a *fundamental role* in inter-universal Teichmüller theory, for instance, in the context of results such as [IUTchIII], Corollary 3.12; in particular, there is no evident analogue of the notion of *log-volume* on finite subextensions of $\mathbb{Q}_p$ contained in ${}^{\dagger}k$, ${}^{\ddagger}k$ that can be **shared** in a consistent fashion between the domain and codomain of isomorphisms as in the diagram of the *second display* of (ii).

§**3.6. Chains of logical AND relations**

From the point of view of the *simple qualitative model* of inter-universal Teichmüller theory given in Example 2.4.5, the discussion of §3.4 concerns the **AND relator "$\wedge$"** in the **"Θ-link"** portion of Example 2.4.5, (ii). On the other hand, strictly speaking, this portion of inter-universal Teichmüller theory only concerns the *initial definition* of the *Θ-link*. That is to say, the *bulk* of the theory developed in [IUTchI-III] concerns, from the point of view of the simple qualitative model of inter-universal Teichmüller theory given in Example 2.4.5, (ii), the **preservation** of the **AND relator "$\wedge$"** as one passes from

- the **"Θ-link"** portion of Example 2.4.5, (ii), to
- the **"multiradial representation"** portion of Example 2.4.5, (ii).

By contrast, the passage from the *"multiradial representation"* portion of Example 2.4.5, (ii), to the *"final numerical estimate"* portion of Example 2.4.5, (ii) — i.e., which corresponds to the passage from [IUTchIII], Theorem 3.11, to [IUTchIII], Corollary 3.12 — is [cf. the discussion of the final portion of Example 2.4.5, (ii)!] *relatively straightforward* [cf. the discussion of §3.10, §3.11, below].

At this point, it is perhaps of interest to consider **"typical symptoms"** of mathematicians who are operating under **fundamental misunderstandings** concerning the essential logical structure of inter-universal Teichmüller theory. Such typical symptoms, which are in fact closely related to one another, include the following:

(Syp1) a sense of **unjustified** and **acutely harsh abruptness** in the passage from [IUTchIII], Theorem 3.11, to [IUTchIII], Corollary 3.12 [cf. the discussion of the final portions of Example 2.4.5, (ii), (iii)!];

(Syp2) a desire to see the "proof" of some sort of **commutative diagram** or **"compatibility property"** to the effect that taking *log-volumes of pilot objects* in the domain and codomain of the Θ-link yields the *same real number* [a property which, in fact, can *never* be proved since it is *false*! — cf. the discussion of §3.5];

(Syp3) a desire to see the *inequality* of the *final numerical estimate* obtained as the result of concatenating some **chain of intermediate inequalities**, i.e., as is often done in proofs in real/complex/functional analysis or analytic number theory.

Here, it should be noted that (Syp2) and (Syp3) often occur as approaches to mitigating the "harsh abruptness" of (Syp1).

With regard to (Syp3), it should be emphasized that it is *entirely unrealistic* to attempt to obtain the *inequality* of the *final numerical estimate* as the result of concatenating some *chain of intermediate inequalities* since this is simply *not* the way in which the logical structure of inter-universal Teichmüller theory is organized. That is to say, in a word, the logical structure of inter-universal Teichmüller theory does *not* proceed by concatenating some sort of chain of intermediate inequalities. Rather,

($\wedge$-Chn) the logical structure of inter-universal Teichmüller theory proceeds by *observing a* **chain** *of* **AND relations** *"$\wedge$"*

[cf. the discussion of [IUTchIII], Remark 3.9.5, (viii), (ix); [IUTchIII], Remark 3.12.2, (c^{itw}), (f^{itw}); [Alien], §3.11, (iv), (v)]. As observed in Example 2.4.5, (ii), (iii), once one follows this *chain of AND relations "$\wedge$"* up to and including the *multiradial representation of the Θ-pilot* [i.e., [IUTchIII], Theorem 3.11], the passage to the *final numerical estimate* [i.e., [IUTchIII], Corollary 3.12] is *relatively straightforward* [i.e., as one might expect, from the use of the word "corollary"!].

One essentially formal consequence of ($\wedge$-Chn) is the following: Since the *definition* of the Θ*-link*, the construction of the *multiradial representation of the* Θ*-pilot*, and the ultimate passage to the *final numerical estimate* consist of a **finite number of steps**, one natural and effective way to **analyze/diagnose** [cf. the discussion of §1.4!] the precise content of **misunderstandings** of inter-universal Teichmüller theory is to determine

($\wedge$-Dgns) **precisely where** in the finite sequence of steps that appear is the **first step** at which the person feels that the **preservation** of the **crucial AND relator** "$\wedge$" is *no longer clear.*

In some sense, the *starting point* of the various *AND relations* "$\wedge$" that appear in the *multiradial algorithm* of [IUTchIII], Theorem 3.11, is the observation that

($\wedge$-Input) the **input data** for this multiradial algorithm consists solely of an **abstract** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip**; moreover, this multiradial algorithm is **functorial** with respect to arbitrary isomorphisms between $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips

[cf. [IUTchIII], Remark 3.11.1, (ii); the final portion of [Alien], §3.7, (i)]. This property ($\wedge$-Input) means that the multiradial algorithm may be applied to *any* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* that appears, or alternatively/equivalently, that *any* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* may serve as the *gluing data* [cf. the "$\gamma_{\mathbb{J}}$" in the analogies discussed in §3.2, (StR3), (StR4), as well as Example 2.4.5, (ii)!] between a *given situation* [i.e., such as the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater in the *codomain* of the Θ-link!] and the *content of the multiradial algorithm.*

On the other hand, in order to conclude that the multiradial algorithm yields **output data** satisfying suitable *AND relations "$\wedge$"*, it is necessary also to examine in detail the content of this output data, i.e., in particular, in the context of the central **IPL** and **SHE** properties discussed in [IUTchIII], Remark 3.11.1, (iii), as well as the **chain of (sub)quotients** aspect of the SHE property [cf. [IUTchIII], Remark 3.11.1, (iii); [IUTchIII], Remark 3.9.5, (viii), (ix)]. In a word, the essential "principle" that is applied throughout the various steps of the multiradial algorithm in order to derive *new AND relations "$\wedge$"* from *old AND relations "$\wedge$"* is the following **"principle of extension of indeterminacies"**:

(ExtInd) If A, B, and C are propositions, then it holds [that $B \implies B \vee C$ and hence] that

$$A \wedge B \implies A \wedge (B \vee C).$$

One important tool that is frequently used in inter-universal Teichmüller theory in a fashion that is closely related to (ExtInd) is the notion of a *poly-morphism* [cf. the discussion of §3.7 below for more details].

In the context of (ExtInd), it is interesting to note that, from the point of view of the discussion of §3.4,

> the "$\vee$" that appears in the conclusion — i.e., $A \wedge (B \vee C)$ — of (ExtInd) may be understood as amounting to essentially the *same phenomenon* as the "$\vee$" that appears in (NeuORInd2) [e.g., by taking "C" to be A].

That is to say, instead of *generating AND relations "$\wedge$" tautologically* by means of the introduction of *distinct labels* [i.e., as in (AOΘ1)] — i.e., say, by introducing a *new distinct label* for "C" so as to conclude a tautological relation

$$A \wedge B \wedge C$$

— (ExtInd) allows one to *generate new AND relations "$\wedge$"* while *avoiding* the introduction of *new distinct labels.* As discussed in §3.4, this point of view [i.e., of avoiding the introduction of new distinct labels] leads inevitably to *OR relations "$\vee$"*, i.e., as in (NeuORInd2) or as in the conclusion "$A \wedge (B \vee C)$" of (ExtInd). As discussed above, the reason that one wishes to **avoid** the introduction of **new distinct labels** when applying (ExtInd) is precisely that

(sQLTL) one wishes to apply (ExtInd) to form **"(sub)quotients/splittings"** of the log-theta-lattice [cf. the title of [IUTchIII]!], i.e., to **project** the *vertical line* on the *left-hand side* of the *infinite "H"* portion of the log-theta-lattice onto the *vertical line* on the *right-hand side* of this infinite "H" by somehow achieving some sort of **"crushing together"** of distinct coordinates [i.e., "(n, m)", where $n, m \in \mathbb{Z}$] of the log-theta-lattice

[cf. the discussion of §3.11 below; [IUTchIII], Remark 3.9.5, (viii), (ix); [IUTchIII], Remark 3.12.2, ($\mathrm{c}^{\mathrm{itw}}$), ($\mathrm{f}^{\mathrm{itw}}$); [Alien], §3.11, (iv), (v)).

At this point, it is of interest to note that there are, in some sense, *two ways* in which (ExtInd) is applied during the execution of the various steps of the multiradial algorithm [cf. the discussion of §3.10, §3.11, below, for more details]:

(ExtInd1) operations that consist of simply adding **more possibilites/indeterminacies** [which corresponds to passing from B to $B \vee C$] within some **fixed container**;

(ExtInd2) operations in which one **identifies** [i.e., *"crushes together"*, by passing from B to $B \vee C$] objects with **distinct labels**, at the cost of passing to a situation in which the object is regarded as being only known **up to isomorphism**.

Typical examples of (ExtInd1) include the *upper semi-continuity of (Ind3)*, as well as the passage to *holomorphic hulls.* Typically, such applications of (ExtInd1) play an important role in establishing various **symmetry** or **invariance** properties such as multiradiality. This sort of establishment of various *symmetry* or *invariance* properties by means of (ExtInd1) then allows one to apply **label crushing** operations as in (ExtInd2). Put another way,

· (ExtInd1) may be understood as a sort of operation whose purpose is to **prepare** suitable **descent data**, while

· (ExtInd2) may be thought of as a sort of actual **descent operation**, i.e., from data that *depends* on the specification of a member of some collection of *distinct labels* to data that is *independent* of such a label specification.

[We refer to the discussion of §3.8 below for more details on *foundational aspects* of (ExtInd2) and to the discussion of §3.9 below for more details concerning the notion of *"descent".*] Typical examples of (ExtInd2) in inter-universal Teichmüller theory are the following [cf. the notational conventions of [IUTchI], Definition 3.1, (e), (f)]:

- identifying "$\Pi_{\underline{v}}$"'s [where $\underline{v} \in \underline{\mathbb{V}}$] at *different vertical coordinates* [i.e., "(n, m)" and "(n, m')", for $n, m, m' \in \mathbb{Z}$] of the log-theta-lattice, which results in a *"$\Pi_{\underline{v}}$ regarded up to isomorphism"* that is labeled by a *new label* "$(n, \circ)$";
- identifying "$G_{\underline{v}}$"'s [where $\underline{v} \in \underline{\mathbb{V}}$] at *different horizontal or vertical coordinates* [i.e., "(n, m)" and "(n', m')", for $n, n', m, m' \in \mathbb{Z}$] of the log-theta-lattice, which results in a *"$G_{\underline{v}}$ regarded up to isomorphism"* that is labeled by a *new label* "$(\circ, \circ)$";
- identifying the *$\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips in the Θ-link* that arise from the Θ- and q-pilot objects in distinct ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theaters [i.e., the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theaters in the *domain* and *codomain* of the Θ-link] by working with these $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips up to isomorphism.

In some sense, the most *nontrivial instances* of the application of (ExtInd) in the context of the multiradial algorithm occur in relation to the **$\mathfrak{log}$-Kummer-correspondence** [i.e., in the *vertical line* on the *left-hand side* of the infinite "H"] and closely related operations of **Galois evaluation** [cf. the discussion of §3.11 below]. The *Kummer theories* that appear in this $\mathfrak{log}$-Kummer-correspondence — i.e., Kummer theories for

- *multiplicative monoids* of nonzero elements of rings of integers in *mixed-characteristic local fields*,
- *mono-theta environments/theta monoids*, and
- *pseudo-monoids of κ-coric functions*

— involve the construction of various **[Kummer] isomorphisms** between

- *Frobenius-like data* and
- *corresponding data constructed via anabelian algorithms from étale-like objects.*

The *output* of such algorithms typically involves constructing the *"corresponding data"* as **one possibility among many**. Here, we note that

> either of these Frobenius-like/étale-like versions of *"corresponding data"* is — unlike, for instance, the data that constitutes an *$\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip*! — *sufficiently robust* that it **completely determines** [even when only regarded up to isomorphism!] the **[usual] embedding** of the **Θ-pilot**.

That is to say, taken as a whole, the **multiradial algorithm** — and, especially, the portion of the multiradial algorithm that involves the $\mathfrak{log}$-*Kummer correspondence* and closely related operations of *Galois evaluation* — plays the role of

> *exhibiting the Frobenius-like Θ-pilot as* **one possibility** *within a* **collection of possibilities** *constructed via anabelian algorithms from étale-like data.*

Thus, in this situation, one obtains the *crucial preservation of the AND relation "$\wedge$"* by applying (ExtInd) twice, i.e., by applying

- (ExtInd1) to the enlargement of the **collection of possibilities** under consideration and
- (ExtInd2) to the **Kummer isomorphisms** involved, when one passes from *Frobenius-like object labels* "(n, m)" [where $n, m \in \mathbb{Z}$] to *étale-like object labels* "$(n, \circ)$" [where $n \in \mathbb{Z}$].

This is precisely what is meant by the **chain of (sub)quotients** aspect of the SHE property [cf. [IUTchIII], Remark 3.11.1, (iii); [IUTchIII], Remark 3.9.5, (viii), (ix)] discussed above [cf. also the discussion of §3.10, §3.11, below].

§3.7. Poly-morphisms and logical AND relations

Poly-morphisms — i.e., sets of morphisms between objects — appear throughout inter-universal Teichmüller theory as a tool for facilitating

> the **explicit enumeration** of a **collection** of **possibilities**.

Composable ordered pairs of poly-morphisms [i.e., pairs for which the domain of the first member in the pair coincides with the codomain of the second member in the pair] may be *composed* by considering the set of morphisms obtained by composing the morphisms that belong to the sets of morphisms that constitute the given pair of poly-morphisms. Such compositions of poly-morphisms allow one to *keep track* — in a *precise* and *explicit* fashion — of *collections* of *possibilities* under consideration.

From the point of view of **chains of AND relations "$\wedge$"**, as discussed in §3.6,

> the **collections** of **possibilities** enumerated by **poly-morphisms** are to be understood as being related to one another via **OR relations "$\vee$"**.

That is to say, poly-morphisms may be thought of as a sort of **indeterminacy**, which is used in inter-universal Teichmüller theory to produce structures that satisfy various **symmetry** or **invariance** properties, hence yield suitable **descent data** [cf. the discussion of (ExtInd1) in §3.6; the discussion of §3.9 below].

Thus, for instance, in the case of the *full poly-isomorphism* that constitutes the Θ-*link*, one may understand the **fundamental AND relation "$\wedge$"** of (AOΘ1) — which, for simplicity, we denote by

$$A \wedge B$$

[where A and B correspond, respectively, in the notation of the discussion of §3.4, to "$* \ :\to \ {}^{\ddagger}\mathfrak{q}\text{-}\mathfrak{plt} \in {}^{\ddagger}\mathfrak{Ring}$" and "$* \ :\to \ {}^{\dagger}\Theta\text{-}\mathfrak{plt} \in {}^{\dagger}\mathfrak{Ring}$"] — may be understood as a relation "$A \wedge (B_1 \vee B_2 \vee \dots)$", i.e., a relation to the effect that

> if one **fixes** the q-pilot ${}^{\ddagger}\mathfrak{q}\text{-}\mathfrak{plt}$, then this q-pilot is **glued**, via the **Θ-link**, to the Θ-pilot ${}^{\dagger}\Theta\text{-}\mathfrak{plt}$ by means of *one isomorphism* [of the full poly-isomorphism that constitutes the Θ-link] or *another isomorphism*, or *yet another isomorphism*, etc.

[Here, the various possible gluings that constitute B are denoted by B_1, B_2, $\dots$.] In particular, as discussed in ($\wedge$-Chn), if one starts with the Θ-*link* and then considers various *subsequent logical AND relations "$\wedge$"* that arise — for instance, by considering various *composites of poly-morphisms*! — by applying (ExtInd), then

($\wedge$($\vee$)-Chn) the *essential logical structure* of inter-universal Teichmüller theory, as discussed in ($\wedge$-Chn), may be understood as follows:

$$\begin{array}{rcl} A \wedge B & = & A \wedge (B_1 \vee B_2 \vee \ldots) \\ & \Longrightarrow & A \wedge (B_1 \vee B_2 \vee \ldots \vee B_1' \vee B_2' \vee \ldots) \\ & \Longrightarrow & A \wedge (B_1 \vee B_2 \vee \ldots \vee B_1' \vee B_2' \vee \ldots \vee B_1'' \vee B_2'' \vee \ldots) \\ & & \vdots \end{array}$$

Finally, we recall that various *"classical examples"* of the notion of a poly-morphism include

- the collection of maps between *topological spaces* that constitutes a *homotopy class*, or *stable homotopy class*, of maps;
- the collection of morphisms between complexes that constitutes a morphism of the associated *derived category*;
- the collection of morphisms obtained by considering some sort of *orbit* by some sort of *group action* on the domain or codomain of a given morphism

[cf. the discussion of [Rpt2018], §13, (PMEx1), (PMEx2), (PMQut)]. Also, in this context, it is useful to recall [cf. the discussion of [Alien], §4.1, (iv)] that

- **gluings via poly-morphisms** are closely related to the sorts of gluings that occur in the construction of **algebraic stacks** [i.e., algebraic stacks which are *not algebraic spaces*].

§3.8. Inter-universality and logical AND relations

One fundamental aspect of inter-universal Teichmüller theory lies in the consideration of **distinct universes** that arise naturally when one considers **Galois categories** — i.e., *étale fundamental groups* — associated to various schemes. Here, it is important to note that, when phrased in this way,

> this fundamental aspect of inter-universal Teichmüller theory is, at least from the point of view of *mathematical foundations*, **no different** from the situation that arises in [SGA1].

On the other hand, the fundamental difference between the situation considered in [SGA1] and the situations considered in inter-universal Teichmüller theory lies in the fact that, whereas in [SGA1], the various distinct schemes that appear are related to one another by means of **morphisms of schemes or rings**,

> the various distinct schemes that appear in inter-universal Teichmüller theory are related to one another, in general, by means of relations — such as the $\mathfrak{log}$- and Θ-*links* — that are **non-ring/scheme-theoretic** in nature, i.e., in the sense that they do **not** arise from morphisms of schemes or rings.

In general, when considering relations between distinct mathematical objects, it is of fundamental importance to specify those mathematical structures that are

common — i.e., in the terminology of inter-universal Teichmüller theory, **coric** — to the various distinct mathematical objects under consideration. Here, we observe that

> this notion of being "common"/"coric" to the various distinct mathematical objects under consideration constitutes, when formulated at a formal, symbolic level, a **logical AND relation "$\wedge$"**.

— cf. the discussion of §3.2 [cf., especially, Example 3.2.2], §3.4, §3.5, §3.6, §3.7.

Thus, in the situations considered in [SGA1], the *ring/scheme structures* of the various distinct schemes that appear are **coric** and hence allow one to relate the universes/Galois categories/étale fundamental groups associated to these distinct schemes in a way that makes use of the *common* ring/scheme structures between these schemes. At a concrete level, this means that

> in the situations considered in [SGA1], étale fundamental groups may be related to one another in such a way that the only *indeterminacies* that occur are **inner automorphism indeterminacies**.

Moreover, these inner automorphism indeterminacies are *by no means superfluous* — cf. the discussion of Examples 3.8.1, 3.8.2, 3.8.3, 3.8.4 below.

<u>Example 3.8.1</u>: Inevitability of inner automorphism indeterminacies. The *unavoidability* of *inner automorphism indeterminacies* may be understood in very elementary terms, as follows.

(i) Let k be a *perfect field*; $\overline{k}$ an *algebraic closure* of k; $N \subseteq G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ a *normal closed subgroup* of G_k; $\sigma \in G_k$ such that the automorphism $\iota_\sigma : N \overset{\sim}{\to} N$ of N given by *conjugating* by σ is *not* inner. [One verifies immediately that, for instance, if k is a *number field* or a *mixed-characteristic local field*, then such N, σ do indeed exist.] Write $k_N \subseteq \overline{k}$ for the subfield of N-invariants of $\overline{k}$, $G_{k_N} \overset{\mathrm{def}}{=} N \subseteq G_k$, $Q_N \overset{\mathrm{def}}{=} G_k/G_{k_N}$. Then observe that this situation yields an example of a situation in which one may verify directly that

> the **functoriality** of the **étale fundamental group** only holds if one allows for **inner automorphism indeterminacies** in the definition of the étale fundamental group.

Indeed, let us first observe that the *"basepoints"* of k and k_N determined by $\overline{k}$ allows us to regard G_k and G_{k_N}, respectively, as the *étale fundamental groups* of k and k_N. Thus, if one assumes that the *functoriality* of the *étale fundamental group* holds *even in the absence of inner automorphism indeterminacies*, then the *commutative diagram of schemes*

$$\begin{tikzcd}
\mathrm{Spec}(k_N) \arrow[rr, "\sigma"] \arrow[dr] & & \mathrm{Spec}(k_N) \arrow[dl] \\
& \mathrm{Spec}(k) &
\end{tikzcd}$$

[where the diagonal morphisms are the natural morphisms] induces a *commutative diagram of profinite groups*

$$\begin{tikzcd}
G_{k_N} \arrow[rr, hook, "\iota_\sigma"] \arrow[dr] & & G_{k_N} \arrow[dl] \\
& G_k &
\end{tikzcd}$$

— which [since the natural inclusion $N = G_{k_N} \hookrightarrow G_k$ is *injective*!] implies that ι_σ is the *identity automorphism*, in contradiction to our assumption concerning σ!

(ii) The phenomenon discussed in (i) may be understood as a consequence of the fact that, whereas $\mathrm{Spec}(k)$ is **coric** in the *commutative diagram of schemes* that appears in (i) [i.e., in the sense that this diagram *does indeed commute*!], $\mathrm{Spec}(\overline{k})$ is **not coric** in the *diagram of schemes*

$$\begin{tikzcd}
& \mathrm{Spec}(\overline{k}) \arrow[dl] \arrow[dr] & \\
\mathrm{Spec}(k_N) \arrow[rr, "\sigma"] \arrow[dr] & & \mathrm{Spec}(k_N) \arrow[dl] \\
& \mathrm{Spec}(k) &
\end{tikzcd}$$

[where the diagonal morphisms are the natural morphisms], i.e., in the sense that the upper portion of this diagram *does not commute*!

(iii) Finally, we consider the *natural exact sequence*

$$1 \longrightarrow G_{k_N} \longrightarrow G_k \longrightarrow Q_N \longrightarrow 1$$

of profinite groups. Then observe that the *inner automorphisms indeterminacies* of G_k [cf. the discussion of (i), (ii)!] induce *outer automorphism indeterminacies* of G_{k_N} that will *not*, in general, be *inner*. That is to say,

> if one considers G_{k_N} in the context of this *natural exact sequence*, then one must in fact consider G_{k_N} [*not only* up to *inner automorphism indeterminacies*, i.e., as discussed in (i), (ii), *but also*] up to certain **outer automorphism indeterminacies**.

Relative to the point of view of the discussion of (ii), these *outer automorphism indeterminacies* may be understood as a consequence of the fact that, in the context of the *field extensions* $k \subseteq k_N \subseteq \overline{k}$ and the *automorphisms* of these field extensions induced by elements of G_k,

the field k is **coric**, whereas the field k_N is **not coric**

— i.e., in the context of these field extensions and automorphisms of field extensions, the relationship of k to the various field extensions that appear is **constant** and **fixed**, whereas the relationship of k_N to the various field extensions that appear is

variable, i.e., subject to **indeterminacies** arising from the action of elements of G_k.

Example 3.8.2: **Inter-universality and the structure of ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theaters.** In the following discussion of *($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theaters*, we fix a collection of *initial Θ-data*

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$$

as in [IUTchI], Definition 3.1, and apply the notational conventions of [IUTchI], Definition 3.1. In particular, we recall that $E \overset{\mathrm{def}}{=} E_F$ is an *elliptic curve* over the number field F; $\overline{F}$ is an *algebraic closure* of F; l is a *prime number*; $K \subseteq \overline{F}$ is the extension field of F determined by the composite of the fields of definition of the closed points of the finite group scheme $E[l] \subseteq E$ of *l-torsion points* of E; $F_{\mathrm{mod}} \subseteq F$ is the *field of moduli* of E, i.e., the field extension of the field of rational numbers generated by the j-invariant of E. For simplicity, we assume that $l > 5$.

(i) We begin by recalling the following:

(i-a) The point of view of **classical Galois theory** with regard to *constructing finite Galois extensions of fields* may be summarized, in the case of the Galois extension K/F, as follows:

- one starts with a *base field* F;
- one then constructs a *finite field extension* K of F that is *saturated* with respect to *Galois conjugation* over F.

Thus, relative to this classical point of view, one is constrained to viewing the situation from the point of view of the *base field* F. This constraint obliges one to always take into account the **entirety** *of Galois conjugates* [over F] of objects associated to K.

The point of view of (i-a) is **fundamentally incompatible** with the *main goal* of the construction of ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theaters in [IUTchI], namely, the **simulation** of a **global multiplicative subspace** of $E[l]$ [cf. the discussion of *global multiplicative subspaces* in [IUTchI], §I1; [Alien], §2.3, §2.4; [Alien], §3.3, (iv), as well as Example 3.2.1, (vi), of the present paper], together with a **global canonical generator**, up to ±1, of the quotient of $E[l]$ by the global multiplicative subspace [cf. the discussion of *global canonical generators* in [IUTchI], §I1; [Alien], §3.3, (iv), as well as Example 3.2.1, (vi), of the present paper]. In some sense, the technical starting point of the *"simulation of a global multiplicative subspace"* implemented in [IUTchI] may be summarized as follows:

(i-b) The *"simulation of a global multiplicative subspace"* given in [IUTchI] is achieved by, in some sense, **reversing** the flow of the classical construction reviewed in (i-a), i.e., by

- viewing the situation [*not* from the point of view of the *base field* F, but rather] from the point of view of the *hyperbolic orbicurve*

$$\underline{C}_K$$

— which may be thought of as data that amounts to K, together with a **fixed choice** of a *quotient "Q"* [cf. [IUTchI], Definition

3.1, (f)] of $E[l]$, i.e., whose *kernel* is to serve as the *"simulated global multiplicative subspace"* — and

- regarding the *base field* F_{mod} — or, at the level of hyperbolic orbicurves, $C_{F_{\mathrm{mod}}}$ [cf. [IUTchI], Remark 3.1.7, (i)] — as a **finite étale quotient** of K [or, at the level of hyperbolic orbicurves, $\underline{C}_K$], i.e., which amounts to thinking in terms of [compatible] *finite étale quotients*

$$\mathrm{Spec}(K) \twoheadrightarrow \mathrm{Spec}(F_{\mathrm{mod}}), \quad \underline{C}_K \to C_{F_{\mathrm{mod}}}$$

— which are regarded as *objects constructed from* $\mathrm{Spec}(K)$, $\underline{C}_K$.

This approach allows one to concentrate on a *fixed [simulated global multiplicative] subspace* and hence [unlike the situation discussed in (i-a)!] to **exclude** the various nontrivial *Galois conjugates* over F of this fixed simulated global multiplicative subspace.

The approach of (i-b) has *numerous important technical consequences* [to be discussed in (ii), (iii), (iv), below].

(ii) From the point of view of *étale-like objects* [i.e., arithmetic fundamental groups], constructing a *quotient* $\underline{C}_K \to C_{F_{\mathrm{mod}}}$ as in (i-b) corresponds to constructing a profinite group "$\Pi_{C_{F_{\mathrm{mod}}}}$" from the profinite group $\Pi_{\underline{C}_K}$ that contains $\Pi_{\underline{C}_K}$ as an open subgroup. In light of the well-known *slimness* of $\Pi_{C_{F_{\mathrm{mod}}}}$ [cf., e.g., [AbsTopI], Proposition 2.3, (ii)], such a construction of "$\Pi_{C_{F_{\mathrm{mod}}}}$" amounts to the construction of a *finite group* of *outer automorphisms* of some open subgroup of $\Pi_{\underline{C}_K}$. This finite group may be thought of as a *finite quotient group* $\Pi_{C_{F_{\mathrm{mod}}}} \twoheadrightarrow \Gamma_{\mathrm{mod}}$ of $\Pi_{C_{F_{\mathrm{mod}}}}$. If we think of the *absolute Galois group* $G_{F_{\mathrm{mod}}}$ of the number field F_{mod} as a *quotient* $\Pi_{C_{F_{\mathrm{mod}}}} \twoheadrightarrow G_{F_{\mathrm{mod}}}$ of $\Pi_{C_{F_{\mathrm{mod}}}}$, then this finite quotient group Γ_{mod} determines a finite quotient group $G_{F_{\mathrm{mod}}} \twoheadrightarrow \Gamma^{\mathrm{Gal}}_{\mathrm{mod}}$ of $G_{F_{\mathrm{mod}}}$. Here, we recall from the construction of [IUTchI], Example 4.3, (i), that $\Gamma^{\mathrm{Gal}}_{\mathrm{mod}}$ has a *natural subquotient* that may be identified with $\mathbb{F}^{*}_{l} \overset{\mathrm{def}}{=} \mathbb{F}^{\times}_{l}/\{\pm 1\}$, i.e., which corresponds to the **multiplicative** $\mathbb{F}^{*}_{l}$**-symmetry** of the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater. In particular, $\Gamma^{\mathrm{Gal}}_{\mathrm{mod}}$, hence also Γ_{mod}, is a finite group of order > 2, which implies, by well-known properties of absolute Galois groups of number fields [cf., e.g., [NSW], Theorem 12.1.7] that

(NoSpl) The surjection $G_{F_{\mathrm{mod}}} \twoheadrightarrow \Gamma^{\mathrm{Gal}}_{\mathrm{mod}}$ of profinite groups does *not* admit a *splitting.*

Here, we note that [in light of the well-known *slimness* of $G_{F_{\mathrm{mod}}}$ — cf., e.g., [AbsTopI], Theorem 1.7, (iii)] this non-existence of a splitting may be reformulated as the assertion that the *natural outer action* of $\Gamma^{\mathrm{Gal}}_{\mathrm{mod}}$ on the kernel $\mathrm{Ker}(G_{F_{\mathrm{mod}}} \twoheadrightarrow \Gamma^{\mathrm{Gal}}_{\mathrm{mod}})$ does *not* admit a lifting to an *action* of $\Gamma^{\mathrm{Gal}}_{\mathrm{mod}}$ on $\mathrm{Ker}(G_{F_{\mathrm{mod}}} \twoheadrightarrow \Gamma^{\mathrm{Gal}}_{\mathrm{mod}})$, i.e., to an action that is *free of inner automorphism indeterminacies.* In particular, it follows [*a fortiori*!] that the *natural outer action* of Γ_{mod} on $\mathrm{Ker}(\Pi_{C_{F_{\mathrm{mod}}}} \twoheadrightarrow \Gamma_{\mathrm{mod}})$ does *not* admit a lifting to an *action* of Γ_{mod} on $\mathrm{Ker}(\Pi_{C_{F_{\mathrm{mod}}}} \twoheadrightarrow \Gamma_{\mathrm{mod}})$, i.e., to an action that is *free of inner automorphism indeterminacies.* That is to say, in summary, the **inner automorphism indeterminacies** in these natural outer actions are **essential** and **unavoidable**.

(iii) The existence of the *inner automorphism indeterminacies* discussed in (ii) implies, in particular, that the *permutations of prime-strips* in the *multiplicative* *symmetry portion* of a ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater induced by the **$\mathbb{F}_l^{*}$-symmetries** of the ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater *necessarily* give rise to *inner automorphism indeterminacies* in the isomorphisms between the copies of *local absolute Galois groups* $G_{\underline{v}}$ [where $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] that appear in prime-strips with *distinct labels* $\in \mathbb{F}_l^{*}$ [cf. [IUTchI], Remark 4.5.1, (iii); [IUTchII], Remark 2.5.2, (iii); [IUTchII], Remarks 4.7.2, 4.7.6; [Alien], §3.6, (iii)]. Put another way,

(NoSyn) there is **no well-defined synchronization** between these copies of $G_{\underline{v}}$ that appear in prime-strips at distinct labels $\in \mathbb{F}_l^{*}$ that is **free** of **inner automorphism** — i.e., **conjugacy** — **indeterminacies**.

In this context, we recall that such a **conjugate synchronization** is of *fundamental importance* in inter-universal Teichmüller theory since it is necessary in order to construct the data that appears in the *unit group portion* of the $\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strip that appears in the *domain* of the Θ*-link*, i.e., data that is required to be **free** of any **dependence** on the distinct labels $\in \mathbb{F}_l^{*}$. Such a *conjugate synchronization* is achieved by applying the **$\mathbb{F}_l^{\rtimes \pm}$-symmetries** [where we recall that $\mathbb{F}_l^{\rtimes \pm} \overset{\mathrm{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\}$, i.e., relative to the natural action of $\{\pm 1\}$ on $\mathbb{F}_l$] in the *additive symmetry portion* of the ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater under consideration [cf. [IUTchII], Corollary 3.5, (i); [IUTchII], Remark 3.5.2, (iii); [IUTchII], Remark 4.5.3, (i); [IUTchIII], Theorem 1.5, (iii); [IUTchIII], Remark 1.5.1, (i); [Alien], §3.6, (ii)]. Here, we observe that in order to achieve this *conjugate synchronization* via the $\mathbb{F}_l^{\rtimes \pm}$*-symmetry* of the various copies of $G_{\underline{v}}$ that appear in prime-strips with distinct labels, it is of fundamental importance to keep these copies of $G_{\underline{v}}$ **isolated** from the **absolute Galois groups** of **number fields** that appear in the discussion of (ii) [i.e., since, as observed in (ii), it is precisely the intrinsic structure of these global absolute Galois groups that gives rise to the *unwanted inner automorphism/conjugacy indeterminacies*!]. This *local-global isolation requirement* — i.e., in effect, the requirement that

(LGIsl) these copies of the *local absolute Galois group* $G_{\underline{v}}$ be regarded **not** as **subgroups** of some global absolute Galois group, but rather as **coric** objects that are treated as being **independent** of any sort of embedding into a *global absolute Galois group*

[cf. [IUTchII], Remark 4.7.6; [Alien], §3.6, (iii)] — will have *important consequences*, as we shall see in the discussion of (iv) below.

(iv) As discussed in (iii), the issue

(SymIsl) of **isolating** the **$\mathbb{F}_l^{\rtimes \pm}$-symmetry** from the **$\mathbb{F}_l^{*}$-symmetry** in order to achieve **conjugate synchronization**

is one important reason for imposing the *local-global isolation requirement* (LGIsl) in the context of the construction of ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters. In fact, however, this property (LGIsl) is fundamental to the *entire structure* of a ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater [cf., [IUTchI], Fig. 6.5; [Alien], Fig. 3.8]. That is to say, the issue (SymIsl) may be thought of as being reflected in the **gluing** along certain collections of prime-strips between the *additive* and *multiplicative* symmetry portions of the ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater [cf., [IUTchI], Fig. 6.5; [Alien], Fig. 3.8]. In fact, however,

(SctNF) even within the *multiplicative symmetry portion* of a ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater, the goal of **simulating** a **global canonical generator** requires

> one to treat the various prime-strips that appear in the multiplicative symmetry portion of the ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater as **"sections"***, in some suitable sense, of the finite étale quotient* $\mathrm{Spec}(K) \twoheadrightarrow \mathrm{Spec}(F_{\mathrm{mod}})$

— a point of view that is *fundamentally incompatible* with the **prime decomposition trees** of the number fields K, F_{mod}, hence again requires one to impose (LGIsl) [cf. [IUTchI], Remarks 4.3.1, 4.3.2; [Alien], §3.3, (iv)]. On the other hand, let us recall that the *ring structure* of the nonarchimedean local field that gives rise to $G_{\underline{v}}$ *cannot be reconstructed* from the *abstract topological group* $G_{\underline{v}}$ [cf. [NSW], the Closing Remark preceding Theorem 12.2.7; [AbsTopIII], §I3; [Alien], Example 2.12.3, (i)]. In particular, once one imposes (LGIsl), the *crucial* reconstruction of the *ring structures* of the nonarchimedean local fields that give rise to the various copies of $G_{\underline{v}}$ — where we recall that such ring structures play a *fundamental* and *indispensable* role in the definition of the **$\mathfrak{log}$-link**! — can only be conducted if one applies the *absolute anabelian algorithms* of [AbsTopIII], §1, *locally* at each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ [not to $G_{\underline{v}}$, but rather] to $\Pi_{\underline{v}}$, i.e., one must always regard each *coric* copy of $G_{\underline{v}}$ as a *"certain quotient"* of a corresponding *coric* copy of $\Pi_{\underline{v}}$. Indeed, *from a historical point of view* [cf. the discussion of [IUTchI], Remark 4.3.2],

> it was precisely these **local-global isolation** aspects — i.e., surrounding (LGIsl), as discussed in (iii) and the present (iv) — of the structure of ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters that motivated the author to develop the **absolute anabelian algorithms** of [AbsTopIII], §1, in the first place!

Example 3.8.3: **Truncated vs. profinite Kummer theory and compatibility with the *p*-adic logarithm.** In the following discussion, we fix notation as follows: Let k be a *finite extension* of $\mathbb{Q}_p$, for some prime number p; $\overline{k}$ an *algebraic closure* of k. Write

- $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$;
- $\mathcal{O}_{\overline{k}}$ for the ring of integers of $\overline{k}$, with maximal ideal $\mathfrak{m}_{\overline{k}} \subseteq \mathcal{O}_{\overline{k}}$;
- $\mathcal{O}^{\triangleright}_{\overline{k}} \subseteq \mathcal{O}_{\overline{k}}$ for the multiplicative monoid of nonzero elements of $\mathcal{O}_{\overline{k}}$;
- $\mathcal{O}^{\times}_{\overline{k}} \subseteq \mathcal{O}^{\triangleright}_{\overline{k}}$ for the group of invertible elements of $\mathcal{O}^{\triangleright}_{\overline{k}}$;
- $\mathcal{O}^{\times}_{\overline{k}} \twoheadrightarrow \mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{k}}$ for the quotient of $\mathcal{O}^{\times}_{\overline{k}}$ by the subgroup μ_∞ of torsion elements [i.e., *roots of unity*] of $\mathcal{O}^{\times}_{\overline{k}}$;
- $\log_{\overline{k}} : \mathcal{O}^{\times}_{\overline{k}} \twoheadrightarrow \overline{k}$ for the *p-adic logarithm* on $\mathcal{O}^{\times}_{\overline{k}}$.

Thus, G_k acts naturally on $\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{k}} \twoheadleftarrow \mathcal{O}^{\times}_{\overline{k}} \subseteq \mathcal{O}^{\triangleright}_{\overline{k}} \subseteq \mathcal{O}_{\overline{k}}$. Let $\Pi \twoheadrightarrow G_k$ be a *topological group* equipped with a surjection onto G_k, which determines natural actions of Π on $\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{k}} \twoheadleftarrow \mathcal{O}^{\times}_{\overline{k}} \subseteq \mathcal{O}^{\triangleright}_{\overline{k}} \subseteq \mathcal{O}_{\overline{k}}$. We shall often think of the pair $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ or the pair $\Pi \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ *"abstractly"* as a pair consisting of an abstract topological monoid [i.e., $\mathcal{O}^{\triangleright}_{\overline{k}}$] equipped with a continuous action by an abstract topological group [i.e., G_k or Π]. Also, we shall write $\mathbb{N}_{\geq 1}$ for the multiplicative monoid of positive natural numbers.

(i) The *three types of Kummer theory* that occur in inter-universal Teichmüller theory [cf. the discussion of Example 3.8.4, (i-a), (i-b), (i-c), below] involve **Kummer towers** that consist of ***N*-th power maps**, for $N \in \mathbb{N}_{\geq 1}$, on the monoids involved. In the case of the pair $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$, one observes immediately that such *N-th power maps* satisfy the following properties [cf. the discussion of [IUTchII], Remark 3.6.4, (i)], where we assume that $N \geq 2$:

(i-a) the *N-th power map* $\mathcal{O}^{\triangleright}_{\overline{k}} \twoheadrightarrow \mathcal{O}^{\triangleright}_{\overline{k}}$ is **not** *a ring homomorphism*, i.e., is **not compatible** with the **additive structure** underlying the **ring structure** on $\mathcal{O}^{\triangleright}_{\overline{k}} \cup \{0\}$;

(i-b) the *N-th power map* $\mathcal{O}^{\triangleright}_{\overline{k}} \twoheadrightarrow \mathcal{O}^{\triangleright}_{\overline{k}}$ is $(\Pi \twoheadrightarrow)$ G_k**-equivariant**, hence may be thought of as inducing on *étale-like cyclotomes* constructed via *anabelian algorithms* from G_k or Π the isomorphism functorially induced by some — at least from an *a priori* point of view — **indeterminate automorphism** [cf. (i-a), which implies that G_k or Π must be treated as *abstract topological groups*, that is to say, as opposed to *groups of ring/field automorphisms*, i.e., *"Galois groups/arithmetic fundamental groups"*; the discussion preceding Example 3.8.1; the discussion following Example 3.8.4 below; the discussion of (vi-c) below] of G_k or Π;

(i-c) the *N-th power map* $\mathcal{O}^{\triangleright}_{\overline{k}} \twoheadrightarrow \mathcal{O}^{\triangleright}_{\overline{k}}$ **alters**, at least from an *a priori* point of view, **cyclotomic rigidity isomorphisms** — but *not synchronizations* between collections of cyclotomic rigidity isomorphisms! — between *étale-like cyclotomes* [cf. (i-b)] and *[Frobenius-like] cyclotomes* arising from the torsion subgroup of $\mathcal{O}^{\triangleright}_{\overline{k}}$ — cf., e.g., the [in general] *nontrivial action* of the N-th power map on the n-th roots of unity in $\mathcal{O}^{\triangleright}_{\overline{k}}$ for $n \in \mathbb{N}_{\geq 1}$ prime to N.

Note that it follows from (i-a) that, if we think of a **"basepoint"** as a **particular "rigid" choice** of an *algebraic closure* that is *free of any conjugacy or N-th power map indeterminacies*, then whereas

(i-d) the *p-adic logarithm*

$$\overline{k} \supseteq \mathcal{O}^{\triangleright}_{\overline{k}} \supseteq \mathcal{O}^{\times}_{\overline{k}} \xrightarrow{\log_{\overline{k}}} \overline{k} \supseteq \mathcal{O}^{\triangleright}_{\overline{k}} \supseteq \mathcal{O}^{\times}_{\overline{k}}$$

yields a precise, well-defined — and, in particular, *free of any conjugacy or N-th power map indeterminacies! — set-theoretic* [but *not ring-theoretic*!] *relationship* between

· the *basepoint* of the ["abstract"] copy of $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ in the *domain* of $\log_{\overline{k}}$ and

· the *basepoint* of the ["abstract"] copy of $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ in the *codomain* of $\log_{\overline{k}}$

[where we think of both of these copies of $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ as *constituent objects* in the respective *Kummer towers* in the domain/codomain of $\log_{\overline{k}}$, as discussed above], i.e.,

a **single unified basepoint**

that is **simultaneously valid** for both the domain and codomain of $\log_{\overline{k}}$, (i-e) the map $\mathrm{Log}_{\overline{k}}$ *induced* by the p-adic logarithm $\log_{\overline{k}}$ on *inverse limits* of the *Kummer tower*

$$\varprojlim_N \ \overline{k} \ \supseteq \ \varprojlim_N \ \mathcal{O}^{\rhd}_{\overline{k}} \ \supseteq \ \varprojlim_N \ \mathcal{O}^{\times}_{\overline{k}} \ \xrightarrow{\mathrm{Log}_{\overline{k}}} \ \overline{k} \ \supseteq \ \mathcal{O}^{\rhd}_{\overline{k}} \ \supseteq \ \mathcal{O}^{\times}_{\overline{k}}$$

[where the inverse limits are over $N \in \mathbb{N}_{\geq 1}$, and we recall that *raising to the N-th power* on the "$\mathcal{O}^{\rhd}_{\overline{k}}$" in the *domain* of $\log_{\overline{k}}$ corresponds to *multiplying by N* on the "$\overline{k}$" in the *codomain* of $\log_{\overline{k}}$] only yields a relationship between

- the *inverse limit basepoint* in the *domain* of $\mathrm{Log}_{\overline{k}}$ and
- the *basepoint* associated to a *single constituent Kummer tower object* [with a *fixed additive structure*!] in the *codomain* of $\mathrm{Log}_{\overline{k}}$ [i.e., "$\mathcal{O}^{\times}_{\overline{k}}$" as opposed to "$\varprojlim_N \ \mathcal{O}^{\times}_{\overline{k}}$"].

Note, moreover, that it follows from (i-b), (i-c) that the **basepoint shifts** that occur as one passes between different *constituent Kummer tower objects* via various *N-th power maps* are indeed — at least from an *a priori* point of view — **substantive/nontrivial** in the context of *cyclotomic rigidity isomorphisms* or *synchronizations between cyclotomes.*

(ii) The situation discussed in (i-e) may be understood as a consequence of the fact that

(ii-a) the *Kummer tower inverse limit* "$\varprojlim$" [where we omit the subscript N to simplify notation] is **biased** toward the **multiplicative structure** of the rings involved — i.e., at the expense of the *additive structures* of these rings [cf. (i-a)] — hence **fundamentally incompatible** with the **"juggling/rotation/permutation"** of the additive and multiplicative structures that arises from the $\mathfrak{log}$-link [cf. the discussion of Example 3.3.2, (iv)].

In the situation of (i-e), we observe, moreover, that the problem of constructing some sort of *"hyper-multiplicative tower"* of copies of, say, the $\varprojlim \ \mathcal{O}^{\times}_{\overline{k}}$ in the *domain* of $\mathrm{Log}_{\overline{k}}$ that *lifts* [i.e., relative to $\mathrm{Log}_{\overline{k}}$] the *[multiplicative] Kummer tower* of copies of $\mathcal{O}^{\times}_{\overline{k}} \subseteq \mathcal{O}^{\rhd}_{\overline{k}} \subseteq \overline{k}$ in the *codomain* of $\mathrm{Log}_{\overline{k}}$ appears to be *unrealistically intractable*: Indeed, *compatibility*, relative to $\mathrm{Log}_{\overline{k}}$, with the [multiplicative] Kummer tower in the codomain of $\mathrm{Log}_{\overline{k}}$ would imply that the *transition maps* of such a *"hyper-multiplicative tower"* would, at least at a *purely formal computational level* (!), necessarily be of the form

$$\begin{aligned} x \ = \ \exp(\log(x)) \ &\mapsto \ \exp(\{\log(x)\}^M) \\ &= \ \exp(\{\log(x)\} \cdot \{\log(x)\}^{M-1}) \ = \ x^{\{\log(x)\}^{M-1}} \end{aligned}$$

— where $M \in \mathbb{N}_{\geq 1}$, and the notation "$\exp(-)$" and "$\log(-)$" is intended in a *purely formal computational sense* (!). On the other hand,

(ii-b) it seems difficult to conceive of any sort of natural approach to constructing such **"hyper-multiplicative transition maps"**

$$\varprojlim \ \mathcal{O}^{\times}_{\overline{k}} \ \to \ \varprojlim \ \mathcal{O}^{\times}_{\overline{k}}$$

that realize the **purely formal computation** "$x \mapsto x^{\{\log(x)\}^{M-1}}$" for $x \in \varprojlim \mathcal{O}^{\times}_{\overline{k}}$ and, moreover, allow one to relate, in some natural way, the [**multiplicative**] **Kummer theory** associated to the $\varprojlim \mathcal{O}^{\times}_{\overline{k}}$ in the **domain** of the transition map to the corresponding [multiplicative] Kummer theory associated to the $\varprojlim \mathcal{O}^{\times}_{\overline{k}}$ in the **codomain** of the transition map.

(iii) Note that the conditions imposed on the *"hyper-multiplicative transition maps"* in (ii-b) are stated in a somewhat rough and imprecise way. Although it is not clear at the time of writing how to make these conditions completely precise, it does, however, seem natural to consider the possibility of the *existence* of *commutative diagrams* as in (iii-a) below, i.e., where one thinks of "Z" as a sort of *candidate* for the *inverse limit* of the "hyper-multiplicative transition maps" of (ii-b). That is to say, the *existence* of such a commutative diagram may be thought of as a sort of *necessary condition* for the existence of a suitable system of "hyper-multiplicative transition maps" as in (ii-b). [Here, we note that the *surjectivity* condition of (iii-a) below may be understood as a sort of "very weak necessary" version of the *domain/codomain Kummer theory-relatability* condition of (ii-b).] In fact, however, we *observe* that, in the situation of (ii-b) [and the surrounding discussion], such a commutative diagram does *not exist*:

(iii-a) Let Z be a set with an action by G_k, $\zeta : Z \to \varprojlim \mathcal{O}^{\times}_{\overline{k}}$ a G_k*-equivariant* map of sets that induces a *surjection* from the J-invariants in the *domain* of ζ to the J-invariants in the *codomain* of ζ for every closed subgroup $J \subseteq G_k$ that acts *trivially* on $\mu_\infty \subseteq \overline{k}$. [Thus, ζ itself is necessarily *surjective*.] Then there does *not* exist any *commutative diagram* of the form

$$\begin{tikzcd}
Z \arrow[r, "\lambda"] \arrow[d, "\zeta"] & \varprojlim \overline{k} \arrow[d, "\psi"] \\
\varprojlim \mathcal{O}^{\times}_{\overline{k}} \arrow[r, "\mathrm{Log}_{\overline{k}}"] & \overline{k}
\end{tikzcd}$$

— where λ is a G_k*-equivariant* map of sets, and ψ is the natural projection.

(iii-b) The map $\mathrm{Log}_{\overline{k}}$ does *not* admit any *factorization*

$$\varprojlim_{N} \mathcal{O}^{\times}_{\overline{k}} \xrightarrow{\lambda} \varprojlim_{N} \overline{k} \xrightarrow{\psi} \overline{k}$$

— where λ is a G_k*-equivariant* map of sets, and ψ is the natural projection.

Indeed, since (iii-b) follows formally from (iii-a), it suffices to verify (iii-a). Suppose that a *commutative diagram* as in (iii-a) exists. Write

- $\mathcal{O}_k \overset{\mathrm{def}}{=} \mathcal{O}_{\overline{k}} \cap k$, $\mathcal{O}^{\times}_k \overset{\mathrm{def}}{=} \mathcal{O}^{\times}_{\overline{k}} \cap k$, $\mathfrak{m}_k \overset{\mathrm{def}}{=} \mathfrak{m}_{\overline{k}} \cap k$;
- $F \overset{\mathrm{def}}{=} k(\mu_\infty) \subseteq \overline{k}$.

Let $x \in p^2 \cdot \mathcal{O}_k$ be a nonzero element. Write

- $f \overset{\mathrm{def}}{=} 1 + x \in \mathcal{O}^{\times}_k$, $(0 \neq)\ y \overset{\mathrm{def}}{=} \log_{\overline{k}}(f) \in \mathfrak{m}_k$;
- $E \subseteq \overline{k}$ for the field extension of F obtained by adjoining all N-th roots of f, for $N \in \mathbb{N}_{\geq 1}$, in $\overline{k}$.

Thus, f lifts [via the natural projection] to an element of the *codomain* of ζ that is *fixed* by the action of $J \overset{\text{def}}{=} \text{Gal}(\overline{k}/E) \subseteq G_k$, hence also [by our *surjectivity* assumption on ζ] to an element $f_Z \in Z$ of the *domain* of ζ that is *fixed* by the action of J. Since λ is G_k*-equivariant*, we thus conclude that $\lambda(f_Z)$ is *fixed* by the action of J and maps via the natural projection ψ to an element $z \in E \subseteq \overline{k}$ that defines a *divisible*, hence l*-divisible* element of $E^\times$, for any prime number $l \neq p$. On the other hand, it follows immediately from the *commutativity* of the *diagram* that $y = z$. Next, observe that since [the *unit*!] f is already clearly l*-divisible* in $k^\times$, hence also in $F^\times$, the Galois group $\text{Gal}(E/F)$ is isomorphic to a quotient of $\mathbb{Z}_p$. But this implies that all l*-power roots* of [the *non-unit*!] $y = z \in k^\times \subseteq F^\times$ are contained in F, in *contradiction* to the easily verified fact that the value group of the valued field F is isomorphic to $\mathbb{Z}[p^{-1}]$. This completes the proof of (iii-a).

(iv) The *qualitatively different behavior* that occurs in (i-d) and (i-e) may be understood as being a consequence of the fact [cf. (i-a)] that whereas

(iv-a) the **ring structure** on $\mathcal{O}^{\triangleright}_{\overline{k}} \cup \{0\}$ [which makes it possible to define the well-known *power series* for the *p-adic logarithm* $\log_{\overline{k}}$] remains intact at any *particular constituent Kummer tower object* [cf. the situation of (i-d)],

(iv-b) the natural multiplicative structure on

$$\varprojlim_N \overline{k} \supseteq \varprojlim_N \mathcal{O}^{\triangleright}_{\overline{k}} \supseteq \varprojlim_N \mathcal{O}^{\times}_{\overline{k}}$$

does **not admit** any corresponding **additive structure** that gives rise to a **ring structure** [i.e., that would make it possible to define the well-known *power series* for the *logarithm*, hence a *factorization* as in (iii-b)] on any of the three inverse limits in the above display that is *compatible* with the *natural action* by G_k and the various *natural projections* to $\overline{k}$ [cf. the situation of (i-e)].

Moreover, we observe that

(iv-c) the various *natural* G_k*-equivariant projections of multiplicative monoids*

$$\varprojlim_N \overline{k} \twoheadrightarrow \overline{k}; \quad \varprojlim_N \mathcal{O}^{\triangleright}_{\overline{k}} \twoheadrightarrow \mathcal{O}^{\triangleright}_{\overline{k}}; \quad \varprojlim_N \mathcal{O}^{\times}_{\overline{k}} \twoheadrightarrow \mathcal{O}^{\times}_{\overline{k}}$$

do **not** admit **splittings** [as may be seen, for instance, by restricting such a splitting to the *roots of unity*, where the existence of such a splitting would amount, in particular, to a splitting of the *natural surjection* $\mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p$].

That is to say, there is *no natural way* to relate the *finite Kummer theory* for a *single constituent Kummer tower object* "$\overline{k}$", "$\mathcal{O}^{\triangleright}_{\overline{k}}$", "$\mathcal{O}^{\times}_{\overline{k}}$" in the *codomain* of the map $\text{Log}_{\overline{k}}$ of (i-e) to the *corresponding profinite Kummer theory* obtained by passing to the *inverse limit* "$\varprojlim_N$" of the associated Kummer tower.

(v) In the context of (iv-b) [and the surrounding discussion], it is also of interest to *observe* that in fact (v-a) *neither* of the inverse limits

$$I_{\overline{k}} \overset{\text{def}}{=} \varprojlim_{N} \overline{k}; \quad I_{\mathcal{O}^{\times}_{\overline{k}}} \overset{\text{def}}{=} \varprojlim_{N} \left(\mathcal{O}^{\times}_{\overline{k}} \cup \{0\} \right)$$

admits a *field structure* that is *stabilized* by the natural action of G_k, and whose underlying *multiplicative structure* is the natural multiplicative structure on the inverse limit.

Indeed, suppose that $I \in \{I_{\overline{k}}, I_{\mathcal{O}^{\times}_{\overline{k}}}\}$ admits such a field structure. Let l be an *odd prime* that does *not divide* the order of the *finite group* of *roots of unity* of k. Write $\mu_l \subseteq \overline{k}$ for the group of *l-th roots of unity* of $\overline{k}$, $(\mu_l \subseteq)\ \mu_{l^\infty} \subseteq \overline{k}$ for the group of *l-power roots of unity* of $\overline{k}$, $K \overset{\text{def}}{=} k(\mu_{l^\infty}) \subseteq \overline{k}$, $G_k \supseteq G_K \overset{\text{def}}{=} \mathrm{Gal}(\overline{k}/K)$, $G_{K/k} \overset{\text{def}}{=} G_k/G_K$, $L \overset{\text{def}}{=} I^{G_K}$ [i.e., the subfield of G_K-invariants of I]. Then our assumption on l implies that the natural faithful action of $G_{K/k}$ on μ_{l^∞} [which allows us to think of $G_{K/k}$ as a *closed subgroup* of $\mathbb{Z}_l^\times$] induces a *nontrivial* action of $G_{K/k}$ on μ_l, hence [in light of the well-known structure of the profinite group $\mathbb{Z}_l^\times \cong \mathbb{Z}_l \times \mathbb{F}_l^\times$ for odd primes l] that $G_{K/k}$ contains a *nontrivial finite closed subgroup* $H \subseteq G_{K/k}$. Next, observe that the group of *divisible elements* of the multiplicative module $K^\times$ is equal to μ_{l^∞} [cf. the fact that $G_{K/k}$ is isomorphic to a closed subgroup of $\mathbb{Z}_l^\times$; [Tsjm], Lemma D, (iii), (iv)]. This implies that the multiplicative $G_{K/k}$-module $L^\times$ is naturally isomorphic to the $G_{K/k}$-module $M_l \overset{\text{def}}{=} \mathrm{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, \mu_{l^\infty}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ [where we note that as an abstract module, M_l is isomorphic to $\mathbb{Q}_l$], hence, in particular, that the field L is of *infinite cardinality*. On the other hand, it follows from *elementary Galois theory* that L is a *finite Galois extension* of the subfield $L^H \subseteq L$ of H-invariants of L. Moreover, since H acts *nontrivially* on μ_l, hence also *nontrivially* on M_l, we thus conclude — from the corresponding fact for the action of nontrivial subgroups of the group of Teichmüller representatives $[\mathbb{F}_l^\times] \subseteq \mathbb{Z}_l^\times$ on $\mathbb{Q}_l$ — that $L^H = \{0, 1\}$ is a set of *cardinality two*, hence that the *infinite field* L is a *finite field*, a contradiction. This completes the proof of (v-a). Note that

(v-b) if it was indeed the case that $I_{\mathcal{O}^{\times}_{\overline{k}}}$ admits a *topological field* structure as in (v-a), then it would be possible to consider the well-known *power series* for the *logarithm* [cf. (iv-a), (iv-b)].

Of course, it follows from (v-a) that such a topological field structure does *not exist*. In particular, the content of (v-a) may be understood as pointing roughly in the same direction as (iii-a), (iii-b), (iv-a), (iv-b).

(vi) Thus, in summary,

(vi-a) the **fundamental dichotomy**, in the context of the *p-adic logarithm*, between

- **[finitely] truncated Kummer theory** [as in (i-d)] and
- **profinite Kummer theory** [as in (i-e)]

may be understood in terms of the **existence** [cf. the "$\overline{k}$" in the *domain* of $\log_{\overline{k}}$ in (i-d)] versus **non-existence** [cf. the *"correspondence"* (Z, ζ, λ) of (iii-a)] of a **single unified basepoint**

for the Kummer theories in the domain/codomain of $\log_{\overline{k}}$ or $\mathrm{Log}_{\overline{k}}$, i.e., a **single set** equipped with an action by G_k that is "sufficiently rich" as to admit *subquotients* [i.e., where we think of λ as in (iii-a) as being *surjective*] that correspond to the *Kummer theories* in the *domain/codomain* of $\log_{\overline{k}}$ or $\mathrm{Log}_{\overline{k}}$.

That is to say,

(vi-b) in the case of *profinite Kummer theory*, the *non-existence* of such a *single unified basepoint* means that one must treat the *Kummer theories* in the *domain/codomain* of $\mathrm{Log}_{\overline{k}}$ — i.e., at a more concrete level, the sets $I_{\overline{k}}$ or $I_{\mathcal{O}^{\times}_{\overline{k}}}$ equipped with their natural multiplicative structures, profinite cyclotomes, and G_k-actions — as being **independent** of one another.

In particular, it follows, essentially formally, from (vi-b) that

(vi-c) one must think of the copies of "G_k" that appear in the *Kummer theories* in the *domain/codomain* of the p-adic logarithm — which may in fact arise as *quotients* of copies of some *topological group* Π in the domain/codomain of the p-adic logarithm — as being related to one another

> **not** as *groups of automorphisms* of the various *monoids* that appear in the Kummer theories in the domain/codomain of the p-adic logarithm, but rather as **abstract topological groups**, i.e., which may be related to one another only by means of some **indeterminate isomorphism** of topological groups $\Pi \overset{\sim}{\to} \Pi$

— cf. the discussion preceding Example 3.8.1, as well as the discussion following Example 3.8.4, concerning the necessity of working, in the context of the $\mathfrak{log}$- and Θ-links, with *abstract topological groups*, that is to say, as opposed to *groups of ring/field automorphisms*, i.e., *"Galois groups/arithmetic fundamental groups"*.

Here, we observe that the *isomorphism indeterminacy* discussed in (vi-c) includes, in particular, *inner automorphisms* of Π. This *chain of observations* (vi-a), (vi-b), (vi-c) forms the *starting point* of the discussion of Example 3.8.4 below.

<u>Example 3.8.4</u>: Symmetrizing isomorphisms, truncatibility, and the $\mathfrak{log}$-Kummer- correspondence. We maintain the notation of Examples 3.8.2, 3.8.3. Also, we shall write "Out$(-)$" for the group of outer automorphisms [i.e., arbitrary automorphisms considered up to inner automorphisms] of a topological group "$(-)$".

(i) In the following discussion, we consider the issue of **compatibility** between

- the various **symmetrizing isomorphisms** arising from the action of the $\mathbb{F}_l^{\circledast}$**-** and $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** on ["abstract"] copies of the pair $G_k \curvearrowright \mathcal{O}^{\rhd}_{\overline{k}}$ [cf. the discussion of Examples 3.8.2, 3.8.3; the theory of [IUTchI], §4, §5, §6; [Alien], §3.3, (v); [Alien], §3.6, (i), (ii), (iii)] and

- the **$\mathfrak{log}$-Kummer-correspondence**

$$\begin{tikzcd}
\dots \arrow[r] & \bullet \arrow[r] \arrow[ddr] & \bullet \arrow[r] \arrow[dd] & \bullet \arrow[r] \arrow[ddl] & \dots \\
\dots & & & & \dots \\
& & \circ & &
\end{tikzcd}$$

[cf. [IUTchIII], Theorem 3.11, (ii)]

for the three types of Kummer theory, namely,

(i-a) the Kummer theory for [“abstract”] copies of the pair $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$, which is based on the classical theory of **Brauer groups/local class field theory** for p-adic local fields [cf. [Alien], §3.4, (v)];

(i-b) the Kummer theory surrounding **theta functions** and **theta values** [cf. [Alien], §3.4, (iii), (iv); [Alien], §3.6, (ii)];

(i-c) the Kummer theory surrounding **κ-coric functions** and copies of the **number field** F_{mod} [cf. the discussion of Example 3.8.2; [IUTchI], Definition 3.1, (b); [Alien], §3.4, (ii); [Alien], §3.6, (iii)]]

that appear in inter-universal Teichmüller theory [cf., e.g., [Alien], Fig. 3.10]. Before proceeding, we recall that the *symmetrizing isomorphisms* arising from the action of the $\mathbb{F}_l^{\circledast}$- and $\mathbb{F}_l^{\rtimes\pm}$-*symmetries* on [“abstract”] copies of the pair $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ **differ** in that

- whereas the symmetrizing isomorphisms arising from the $\mathbb{F}_l^{\rtimes\pm}$-*symmetries* are *free of inner automorphism indeterminacies* and hence give rise to **conjugate synchronizations**,
- the symmetrizing isomorphisms arising from the $\mathbb{F}_l^{\circledast}$-*symmetries* necessarily involve **inner automorphism indeterminacies**

[cf. the discussion of Example 3.8.2]. On the other hand, it follows immediately — i.e., by considering the symmetrizing isomorphisms induced by arbitrary elements of $\mathrm{Aut}(\underline{X}_K) \xrightarrow{\sim} \mathrm{Out}(\Pi_{\underline{X}_K})$ [cf. the notation of [IUTchI], Definition 3.1, (d); [Alien], §3.3, (v)] — that

> if one *forgets* about the issue of *conjugate synchronization* and just thinks in terms of *arbitrary [indeterminate] isomorphisms* between [“abstract”] copies of the pair $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$, then the *symmetrizing isomorphisms* arising from the action of the $\mathbb{F}_l^{\circledast}$- and $\mathbb{F}_l^{\rtimes\pm}$-*symmetries* on [“abstract”] copies of the pair $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ in fact **coincide**.

(ii) In the case of the *Kummer theory* of (i-a),

(ii-a) **compatibility** between the **$\mathbb{F}_l^{\rtimes\pm}$-symmetrizing isomorphisms** — i.e., *without conjugacy indeterminacies*! [cf. the final portion of (i)] — and the **Kummer theories** of (i-a) in the *domain/codomain* of the **$\mathfrak{log}$-link** then follows formally by applying *transport of structure* via the **$\mathbb{F}_l^{\rtimes\pm}$-symmetries** to the **truncated** *Kummer theories* in the domain/codomain of the $\mathfrak{log}$-*link*, computed relative to the **single unified basepoint** discussed in Example 3.8.3, (i-d), (vi-a), at each *evaluation label* "$t \in \mathbb{F}_l$"

[cf. the discussion of [IUTchIII], Remark 2.3.3, (viii); [Alien], §3.6, (ii)]. Here, we recall that

(ii-b) this *compatibility* is necessary to define the **diagonal** "$0/\succ/>$" — i.e., with respect to the *evaluation labels* "$t \in \mathbb{F}_l$" — *local data* that is used to construct the *local unit group* [i.e., "$\mathcal{O}^{\times \boldsymbol{\mu}}$"] portion of the **gluing data** that appears in the **Θ-link**

[cf. [IUTchIII], Theorem 1.5, (iii); [IUTchIII], Remark 1.5.1, (i); [Alien], §3.6, (ii)]. In this context, it is important to recall that

(ii-c) this *local unit group* portion of the *Θ-link gluing data* satisfies the crucial property of being **independent** of the *evaluation labels* "$t \in \mathbb{F}_l$" — which are only well-defined **internally** within a *particular* (*$\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater*! — hence allows one to construct the **containers** [that is to say, in the form of *tensor-packets* of *log-shells*] that appear in the **multiradial representation** of the **Θ-pilot**, i.e., the containers that make it possible to represent the Θ-pilot in the *domain* (*$\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater* of the Θ-link in terms of **"external"** data arising from the *codomain* (*$\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater* of the Θ-link.

Put another way,

(ii-d) if it were the case that the *containers* of (ii-c) could only be constructed from the *domain* (*$\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater* of the Θ-link in a way that involves **independent conjugacy indeterminacies**

- at the *distinct evaluation labels* "$t \in \mathbb{F}_l$" [cf. (ii-b), (ii-c)] or
- in the *domain/codomain* of the $\mathfrak{log}$-*link* [cf. (ii-a)]

— i.e., all of which are only well-defined **internally** within the *domain* (*$\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater* of the Θ-link! — then it would follow that these containers **cannot** be constructed in a way that is well-defined **externally** to the *domain* (*$\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater* of the Θ-link, e.g., in terms of data arising from the *codomain* (*$\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater* of the Θ-link.

In particular, we observe that the **truncatibility** of the Kummer theory of (i-a) plays a *fundamental role* in the logical structure of inter-universal Teichmüller theory.

(iii) The discussion of (ii-b), (ii-c), (ii-d) centers on the issue of *synchronizing the conjugacy indeterminacies* at the *diagonal label* "$\succ / >$" in the *domain/codomain* of the $\mathfrak{log}$-*link*. This focus of attention on the diagonal label thus prompts the following question:

(iii-a) If one is only interested in *synchronizing the conjugacy indeterminacies* at the **diagonal label** "$\succ / >$" in the *domain/codomain* of the $\mathfrak{log}$-*link*, then why does it not suffice to relate the **profinite** [i.e., rather than *truncated*!] versions of the **Kummer theory** of (i-a) for the *diagonal label* "$\succ / >$" in the *domain/codomain* of the $\mathfrak{log}$-*link* via a *single* [i.e., corresponding to the *"single"* diagonal label] *indeterminate isomorphism* "$\Pi \xrightarrow{\sim} \Pi$" as in Example 3.8.3, (vi-c)?

In fact, however, the approach described in (iii-a) is *not sufficient* for the following reason:

(iii-b) Ultimately, in inter-universal Teichmüller theory, one is interested in constructing the **multiradial representation** of the **Θ-pilot** [cf. [IUTchIII], Theorem 3.11] — which involves the **theta values**

$$``\underline{\underline{q}}_{\underline{v}}^{j^2}"$$

at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ — i.e., whose construction depends, in an essential way, on the use of **[global] independent** labels "$t \in \mathbb{F}_l$" [cf. (ii-d)] at which the **conjugacy indeterminacies** are nevertheless **synchronized** relative to a **single basepoint** arising from the *global additive symmetry portion* "$\mathcal{D}^{\circledcirc\pm}$" of the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater under consideration [cf. [IUTchI], Definition 6.1, (v); [IUTchII], Corollary 4.5, (iii), (iv); [IUTchII], Corollary 4.6, (iii), (iv)].

That is to say,

(iii-c) the requirements discussed in (iii-b) are satisfied by the approach that is *actually* taken in inter-universal Teichmüller theory [cf. (ii-a); [IUTchIII], Proposition 1.3, (i); [IUTchIII], Remark 1.3.2], i.e., of *synchronizing the conjugacy indeterminacies* in the *domain/codomain* of the $\mathfrak{log}$*-link* **locally** *"label by label"*, for the various labels "$t \in \mathbb{F}_l$" [cf. (ii-d)], so that arbitrary conjugacy indeterminacy synchronizations in the *domain* of the $\mathfrak{log}$*-link* are **reflected faithfully** in the *codomain* of the $\mathfrak{log}$*-link*.

On the other hand,

(iii-d) the approach discussed in (iii-c) can be implemented precisely because of the existence of the canonical **single unified basepoints** for the *domain/codomain* of the $\mathfrak{log}$*-link* discussed in Example 3.8.3, (vi-a), i.e., which are available only in the case of **truncated Kummer theory**, since the *profinite Kummer theory conjugacy indeterminacies* of Example 3.8.3, (vi-c), would give rise [in the situation of the approach discussed in (iii-c)] to *independent conjugacy indeterminacies* at the various labels "$t \in \mathbb{F}_l$".

Thus, in summary, the discussion of the present (iii) sheds further light on the *fundamental role* played by the **truncatibility** of the Kummer theory of (i-a) in the logical structure of inter-universal Teichmüller theory.

(iv) In the case of the *Kummer theory/Galois evaluation* of (i-c), let us first recall from the discussion of [IUTchIII], Remark 2.3.3, (vi), (vii) [cf. also [Alien], §3.4, (ii)] that

(iv-a) the fact that the *submonoid*

$$\mathbb{I}^{\mathrm{ord}} \subseteq \mathbb{N} \times \{\pm 1\}$$

generated by the set of **orders** of the **zeroes/poles** [considered as *signed* elements of $\mathbb{N} \times \{\pm 1\}$] of the **rational functions** that appear in (i-c) contains — i.e., unlike the case with the **theta functions** that appear in (i-b)! — elements $\notin \{\pm 1\}$, as well as elements $\notin \mathbb{N}$, means that the **cyclotomic rigidity isomorphisms** obtained in (i-c) may only be constructed in a fashion consistent with the **anabelian reconstruction algorithms** of [AbsTopIII], Theorem 1.9 [cf. also [IUTchI], Remark 3.1.2, (ii), (iii)] if one constructs these *cyclotomic rigidity isomorphisms*

- · via **profinite Kummer theory** [i.e., by applying the fact that $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\}$] and
- · up to an **indeterminacy** given by multiplication by elements of the image $\mathbb{I}^{\text{ord}}_{\pm} = \{\pm 1\}$ of the projection $\mathbb{I}^{\text{ord}} \twoheadrightarrow \{\pm 1\}$ to the second factor.

— i.e., in particular, relative to the constraints discussed in Example 3.8.3, (vi-b), (vi-c).

That is to say, it follows from the discussion of Example 3.8.3, (vi-b), (vi-c) — cf. also

- · the **splitting/decoupling** of the **unit group** portion from the *pseudo-monoid of κ-coric rational functions* [as discussed in [IUTchI], Example 5.1, (v); [Alien], §3.4, (ii)];
- · the **non-interference** properties satisfied by the Frobenius-like copies of $F^\times_{\text{mod}}$ in the *domain/codomain* of the $\mathfrak{log}$-*link* [as discussed in [IUTchIII], Proposition 3.10, (ii); [Alien], §3.7, (i)]

— that

(iv-b) the *Kummer theories/Galois evaluation operations* of (i-c) in the *domain/codomain* of the $\mathfrak{log}$-*link*

- · involve **[pseudo-]monoids** that must be treated **independently** of one another, hence, in particular,
- · may be related to one another only up to **indeterminacies** that involve *indeterminate isomorphisms* between corresponding Galois groups/arithmetic fundamental groups in the *domain/codomain* of the $\mathfrak{log}$-*link*

[cf., especially, the discussion of the final portion of Example 3.8.3, (vi)].

On the other hand,

(iv-c) the **compatibility** between the $\mathbb{F}^{\circledast}_l$**-symmetrizing isomorphisms** — i.e., *with conjugacy indeterminacies*! [cf. the discussion of the final portion of (i)] — and the **Kummer theories/Galois evaluation operations** of (i-c) in the *domain/codomain* of the $\mathfrak{log}$**-link** follows formally by applying *transport of structure* via the $\mathbb{F}^{\circledast}_l$**-symmetries** to the **profinite** *Kummer theories* of (i-c) in the domain/codomain of the $\mathfrak{log}$-link, together with the *compatibility* of the $\mathfrak{log}$-*link* with these $\mathbb{F}^{\circledast}_l$-*symmetrizing isomorphisms* [cf. [IUTchIII], Proposition 1.3, (i), (ii); [IUTchIII], Remark 1.3.3, (ii)].

As a result of the various *indeterminacies* of (iv-b), (iv-c),

(iv-d) in the case of the *Kummer theories/Galois evaluation operations* of (i-c), the only **diagonal** "$0/ \succ / >$" — i.e., with respect to the *evaluation* *labels* "$j \in \mathbb{F}_l^{\circledast}$" — **number field** inside the algebraic closure $\overline{F}$ that is well-defined **externally** to the *domain (*$\Theta^{\pm\mathrm{ell}}$*NF-)Hodge theater* of the Θ*-link* [cf. the discussion of the *multiradial representation* in (ii-c), (ii-d)] is F_{mod} [cf. [Alien], §3.6, (iii)].

Here, we observe that

(iv-e) the $\mathbb{I}^{\mathrm{ord}}_{\pm}$*-indeterminacies* of (iv-a) may be **synchronized**

- relative to the $\mathbb{F}_l^{\circledast}$**-symmetry** by applying the **"linear disjointness"** of the theory of κ-coric functions from $SL_2(\mathbb{F}_l)$ [cf. the second display of [Alien], §3.6, (iii)];
- relative to the distinct **valuations** $\underline{v} \in \underline{\mathbb{V}}$ in *local-global comparisons* of the *Kummer theories/Galois evaluation operations* of (i-c) by means of comparison with the *cyclotomic rigidity isomorphisms* arising from the *Kummer theories of (i-a)* [i.e., which are *not subject* to $\mathbb{I}^{\mathrm{ord}}_{\pm}$*-indeterminacies*]

[cf. the discussion of the final portion of [IUTchIII], Remark 2.3.3, (vi)], but

- **not** relative to the *domain/codomain* of the $\mathfrak{log}$**-link** since the relevant *[Frobenius-like] monoids* in the domain/codomain of the $\mathfrak{log}$-link are not set-theoretically related to one another via the $\mathfrak{log}$-link

[cf. (iv-b)].

In particular, we note that one *fundamental aspect* of the Kummer theory/Galois evaluation of (i-c) that underlies the *compatibility* and *synchronization* properties of (iv-c), (iv-e) is the $\mathbb{F}_l^{\circledast}$**-symmetricity** of the Kummer theory/Galois evaluation of (i-c) [cf. the discussion of [IUTchII], Remark 1.1.1, (v); the first display of [IUTchIII], Remark 2.3.3, (iii); the discussion of [Alien], §3.6, (iii)].

(v) The Kummer theory/Galois evaluation of (i-b), which does **not** satisfy the condition of $\mathbb{F}_l^{\rtimes\pm}$**-symmetricity**, exhibits *qualitatively fundamentally different* behavior from the $\mathbb{F}_l^{\circledast}$*-symmetric* Kummer theory/Galois evaluation of (i-c) [cf. the discussion of [IUTchII], Remark 1.1.1, (v); the first display of [IUTchIII], Remark 2.3.3, (iii)]. If, for instance,

(v-a) one attempts to apply the approach via **profinite Kummer theory** of (iv-b), (iv-c) to the task of verifying some sort of **compatibility** between the $\mathbb{F}_l^{\rtimes\pm}$**-symmetrizing isomorphisms** — i.e., with **conjugacy indeterminacies** arising from $\mathfrak{log}$*-link domain/codomain comparisons*, as discussed in Example 3.8.3, (vi-b), (vi-c) [cf. also the discussion of (i), (ii), (iii), especially (iii-d), of the present Example 3.8.4]! — and the *Kummer theories/Galois evaluation operations* of (i-b) in the *domain/codomain* of the $\mathfrak{log}$*-link*,

then one encounters the following situation:

(v-b) In order to **compare**, via the $\mathbb{F}_l^{\rtimes\pm}$*-symmetry*, the Kummer theories/Galois evaluation operations at *different* $t \in \mathbb{F}_l$ [i.e., for *distinct theta values*

"$\underline{\underline{q}}_{\underline{v}}^{j^2}$", where $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], it is necessary to establish a(n) — *a priori* **self-contradictory**!— situation in which the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* permutes the **labels** "$t \in \mathbb{F}_l$" in a **nontrivial** fashion, but acts **trivially** on the [*non-*$\mathbb{F}_l^{\rtimes\pm}$*-symmetric*!] **étale theta function**!

Here, we recall [cf. the theory of [IUTchII], §2] that

(v-c) in the context of Galois evaluation, the *étale theta function* may be thought of as a cohomology class

$$\in H^1\left(\begin{array}{c}\text{some \textit{closed subgroup}}\\ \text{of } \Pi_{\underline{v}}\end{array}, \quad \begin{array}{c}\text{some \textit{cyclotome} associated to the}\\ \text{Galois evaluation label } t \in \mathbb{F}_l\end{array}\right)$$

— that is to say, where [cf. (v-b)!] the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* is to **act nontrivially** on the **Galois evaluation cyclotome** [i.e., the *second* argument of "$H^1(-)$"], but **trivially** on the **closed subgroup of** $\Pi_{\underline{v}}$ [i.e., the *first* argument of "$H^1(-)$"]!

Before proceeding, it is also of interest to recall that

(v-d) a *profinite Kummer theory* situation of the sort discussed in (v-a), (v-b), (v-c) occurs, for instance, if one replaces the theta functions that occur in inter-universal Teichmüller theory by their ***N*-th powers**, where N is an integer ≥ 2 [cf. the discussion of [IUTchIII], Remark 2.3.3, (vi), (vii)].

Now returning to the discussion of (v-b), (v-c), we observe that

(v-e) it is essentially a *tautology* that the *only "solution"* to the *[a priori] self-contradictory* **simultaneous "trivial/nontrivial action" condition "(SimCon)"** of (v-b), (v-c) lies in working with objects that are **invariant** with respect to the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, i.e., at a more concrete level, with a

single unified set-theoretic basepoint

— which allows one to compute the various $\mathbb{F}_l^{\rtimes\pm}$-*symmetrizing isomorphisms* by projecting to the single copy of "$G_{\underline{v}}$" determined by the basepoint.

On the other hand,

(v-f) the existence of the **conjugacy indeterminacies** inherent in the *profinite Kummer theory* $\mathbb{F}_l^{\rtimes\pm}$-*symmetrizing isomorphisms* of (v-a) means that any *"single unified basepoint"* as in (v-e) is well-defined only up to *conjugacy indeterminacies* — a situation that is *unacceptable* in inter-universal Teichmüller theory since it would mean that the *coefficient cyclotomes* in (v-c) are well-defined only up to certain $\widehat{\mathbb{Z}}^\times$-*multiples*, i.e., that the **étale theta functions** and **theta values** [i.e., "$\underline{\underline{q}}_{\underline{v}}^{j^2}$", where $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] in the theory are well-defined only up to certain $\widehat{\mathbb{Z}}^\times$**-powers** [cf. the discussion of [IUTchIII], Remark 2.1.1, (v)]!

That is to say,

(v-g) the only way to avoid the pathologies of (v-f) is to *replace* the profinite Kummer theory in the discussion of (v-a), (v-b), (v-c), (v-e), (v-f) by the **truncated Kummer theory** of (i-b) [cf. the discussion of the final portion of [Alien], §3.6, (ii)], so that we can apply the *single unified basepoint* of Example 3.8.3, (i-d), (vi-a) [cf. also the discussion of (ii) in the present Example 3.8.4] to obtain a

single unified set-theoretic basepoint

as in (v-e), but which is well-defined up to **geometric fundamental group conjugacy indeterminacies** [i.e., indeterminacies arising from conjugation by elements of the geometric fundametric fundamental groups involved], which are, at any rate, inherent in the $\mathbb{F}_l^{\rtimes\pm}$-*symmetrizing isomorphisms* and, moreover, [unlike the situation discussed in (v-f)!] *do not have any effect* on the computation of these $\mathbb{F}_l^{\rtimes\pm}$-*symmetrizing isomorphisms* by projecting to the single copy of "$G_{\underline{v}}$" determined by the basepoint

— cf. [IUTchII], Remark 1.1.1, (iv), (v); [IUTchII], Remark 2.6.1, (i), (ii). Before proceeding, it is interesting to observe that

(v-h) the *"tautological resolution"* of the *[a priori] self-contradictory* **simultaneity condition (SimCon)** discussed in (v-e), (v-g) by considering *"invariants"* — i.e., in the situation of (v-e), (v-g), a *single unified set-theoretic basepoint* — is *formally highly reminiscent* of the *"tautological resolution"* of the **$\mathfrak{log}$-shifts** in the left- and right-hand columns of the *"infinite H"* of (InfH) by means of the construction of **invariants** [cf. the discussion surrounding ($\mathfrak{log}$ORInd), (Di/NDi) in §3.11 below; the discussion of (Stp7) in §3.10 below].

Thus, in summary, the **truncatibility** of the *Kummer theory/Galois evaluation operations* of (i-b) [cf. the discussion of [IUTchIII], Remark 2.3.3, (vi), (vii), (viii); the final portion of [Alien], §3.6, (ii)] plays a *fundamental role* in the logical structure of inter-universal Teichmüller theory [cf. the discussion of Example 3.3.2, (vii)].

(vi) Recall that the **single unified basepoint** of (v-g) is completely determined by the **connected subgraph** "$\Gamma^{\blacktriangleright}_{\underline{\underline{X}}} \subseteq \Gamma_{\underline{\underline{X}}}$", or, equivalently, the connected subgraph "$\Gamma^{\blacktriangleright}_{\ddot{Y}} \subseteq \Gamma_{\ddot{Y}}$" [cf. [IUTchII], Remark 2.6.1, (i), (ii); [IUTchII], Remark 2.6.3, (i)]. Here, we recall further from [IUTchII], Remark 2.6.3, (i), that $\Gamma_{\underline{\underline{X}}}$ or $\Gamma_{\ddot{Y}}$ may be thought of as a *"copy* Γ *of the real line* $\mathbb{R}$*"*, in which the integers $\mathbb{Z} \subseteq \mathbb{R}$ are taken to be the *vertices*, and the line segments joining the integers are taken to be the *edges*. In light of the *central role* played by the *singled unified basepoint* of (v-g) in the discussion of (v) — and indeed in the **entire logical structure** of inter-universal Teichmüller theory! — it is of interest to recall from the discussion of [IUTchII], Remark 2.6.3, that this *subgraph* "$\Gamma^{\blacktriangleright}$" is [essentially] **uniquely determined** by various natural conditions, which must be satisfied in order for the theory to operate in the desired fashion. Indeed, let us first recall from the discussion of [IUTchII], Remark 2.5.2, (i), (ii), (iii), (iv); [IUTchII], Remark 2.6.3, (ii), that

(vi-a) the various *geometric fundamental groups* "Δ" that appear act on the various *vertices* $t \in \{-l^*, \dots, -1, 0, 1, \dots, l^*\} \subseteq \Gamma$ **independently**

— where we recall that $l^{\circledast} = \frac{l-1}{2}$, and that

(vi-b) the actions referred to in (vi-a) of *geometric fundamental groups* "Δ" on the *"copy* Γ *of the real line* $\mathbb{R}$*"* amount to the conventional action of $G_\Gamma \overset{\mathrm{def}}{=} l \cdot \mathbb{Z} \rtimes \{\pm 1\}$ [i.e., the group generated by *translations* by elements of $l \cdot \mathbb{Z}$ and *multiplication* by -1] on the real line $\mathbb{R}$.

Write $\Gamma^{\blacktriangleright}_{\mathbb{Z}} \overset{\mathrm{def}}{=} \Gamma^{\blacktriangleright} \cap \mathbb{Z}$; $\pm\Gamma^{\blacktriangleright}_{\mathbb{Z}} \overset{\mathrm{def}}{=} \Gamma^{\blacktriangleright}_{\mathbb{Z}} \cup -\Gamma^{\blacktriangleright}_{\mathbb{Z}} \subseteq \mathbb{R}$. Next, recall that

(vi-c) since the **vertex** $0 \in \Gamma$ is the *unique vertex* in Γ that may be used to define the crucial **splittings** of **theta monoids** discussed in [IUTchII], Corollary 2.6, (ii), it is necessary that $0 \in \Gamma^{\blacktriangleright}_{\mathbb{Z}}$ [cf. [IUTchII], Remark 2.6.3, (i)].

On the other hand, (vi-c), together with the existence of the *independent* G_Γ-*indeterminacies* of (vi-a), (vi-b), imply that

(vi-d) if $\Gamma^{\blacktriangleright}$ contains a *connected component* $\Gamma^{\blacktriangleright}_{*}$ that is *not* contained in $l \cdot \mathbb{Z}$ ($\subseteq \Gamma$), then any **theta value**

$$\underline{\underline{q}}_{\underline{v}}^{j^2}$$

obtained via **Galois evaluation** for $j \in \Gamma^{\blacktriangleright}_{*} \cap \mathbb{Z} \subseteq \Gamma$, is *well-defined only up to a* G_Γ*-indeterminacy*, i.e., up to a *possible confusion* between j and G_Γ*-translates* j' of j.

(vi-e) any **theta value**

$$\underline{\underline{q}}_{\underline{v}}^{j^2}$$

obtained via **Galois evaluation** for $j \in \Gamma^{\blacktriangleright}_{\mathbb{Z}}$, is *well-defined only up to a* G_Γ*-indeterminacy within* $\Gamma^{\blacktriangleright}_{\mathbb{Z}}$, i.e., up to a *possible confusion* between j and G_Γ*-translates* j' of j that lie inside $\Gamma^{\blacktriangleright}_{\mathbb{Z}}$.

Here, it is important to recall [cf. [IUTchI], Remark 3.5.1, (ii); [IUTchII], Remark 2.6.3, (iv); [IUTchII], Corollary 4.5, (v); [IUTchII], Corollary 4.6, (v)] that

(vi-f) any *indeterminacies concerning "j", "j'"* of the sort described in (vi-d), (vi-e) for nonzero $j, j' \in \mathbb{Z}$ with *distinct absolute values* would result in indeterminacies in the *"vector of ratios"* [cf. [IUTchII], Corollary 4.5, (v)] that determines the structure of the *global realified Gaussian Frobenioids*, i.e., would result in **violations** of the **global product formula** relating the value groups at different $\underline{v} \in \underline{\mathbb{V}}$.

In particular, (vi-c), (vi-d), (vi-e), (vi-f) imply that

(vi-g) the *natural map*

$$\pm\Gamma^{\blacktriangleright}_{\mathbb{Z}}/\{\pm 1\} \to \mathbb{Z}/G_\Gamma$$

— i.e., from the set of $\{\pm 1\}$-orbits of $\pm\Gamma^{\blacktriangleright}_{\mathbb{Z}}$ to the set of G_Γ-orbits in $\mathbb{Z}$ — is **injective** [cf. [IUTchII], Remark 2.6.3, (ii)];

(vi-h) the **subgraph** $\Gamma^{\blacktriangleright} \subseteq \Gamma$ is **connected** [cf. [IUTchII], Remark 2.6.3, (i)].

Now one verifies immediately that (vi-c), (vi-g), (vi-h) imply *formally* that

(vi-i) $\Gamma^{\blacktriangleright} \subseteq \Gamma$ is a **connected subgraph** that contains 0 and is **contained** in the closed interval $[-l^{\circledast}, l^{\circledast}]$.

Next, we observe that, in light of (vi-e), (vi-i),

(vi-j) replacing $\Gamma^{\blacktriangleright}_{\mathbb{Z}}$ by $\pm\Gamma^{\blacktriangleright}_{\mathbb{Z}}$ does *not* have any effect on the validity of (vi-i) or on the way in which the subgraph $\Gamma^{\blacktriangleright}_{\mathbb{Z}}$ is used in inter-universal Teichmüller theory; in particular, we may *assume*, without loss of generality, that the **symmetry** condition $\Gamma^{\blacktriangleright}_{\mathbb{Z}} = \pm\Gamma^{\blacktriangleright}_{\mathbb{Z}}$ holds.

On the other hand,

(vi-k) the *estimates* that are *ultimately* obtained in inter-universal Teichmüller theory [cf. [IUTchIII], Corollary 3.12; [IUTchIV], Theorem 1.10] are **optimized** precisely when the *average* of the *squares* j^2 of the nonzero elements $j \in \Gamma^{\blacktriangleright}_{\mathbb{Z}} = \pm\Gamma^{\blacktriangleright}_{\mathbb{Z}}$ is **maximized** [cf. [IUTchII], Remark 2.6.3, (ii)], i.e., when $\Gamma^{\blacktriangleright} = \pm\Gamma^{\blacktriangleright}_{\mathbb{Z}}$ is given precisely by the **closed interval** $[-l^*, l^*]$ [which implies that the *natural map* of (vi-g) is **bijective**].

That is to say, in summary, when subject to the *symmetry* condition $\Gamma^{\blacktriangleright}_{\mathbb{Z}} = \pm\Gamma^{\blacktriangleright}_{\mathbb{Z}}$ of (vi-j) and the *optimization* condition of (vi-k), the subgraph $\Gamma^{\blacktriangleright} \subseteq \Gamma$ is in fact **uniquely determined** [cf. [IUTchII], Remark 2.6.3, (v)].

Unlike the situations considered in [SGA1] [cf. the discussion of Example 3.8.1], in which the *ring/scheme structures* of the various distinct schemes that appear are **coric**, the *ring structures* of the rings that appear on either side of the $\mathfrak{log}$- and Θ-links of inter-universal Teichmüller theory — i.e., such as number fields or completions of number fields at various valuations — are **not coric** with respect to the respective links. This leads one naturally to consider **weaker structures** [cf. the discussion of Example 3.2.2, (i), (ii), (iv)] the discussion of Example 3.8.2, (iii), (iv); the discussion of Example 3.8.3, (vi)] such as

- *abstract topological groups*, in the case of the *profinite Kummer theories* in the *domain/codomain* of the $\mathfrak{log}$-*link*, or
- *sets* equipped with a *topology* and a continuous action of a *topological group*, in the case of the $\mathfrak{log}$-*link*, or
- *realified Frobenioids* [in the sense of [IUTchIII], Theorem 1.5, (v)] or *topological monoids* equipped with a continuous action of a *topological group*, in the case of the Θ-*link*,

which are indeed **coric** with respect to the respective links. Indeed, it is precisely this sort of consideration — i.e., of *weaker coric structures* to relate the *universes/Galois categories/étale fundamental groups* associated to ring/scheme structures on opposite sides of the links under consideration [cf. the discussion preceding Example 3.8.1] — that gave rise to the term **"inter-universal"**.

Here, we note that it is of fundamental importance that these *topological groups* [which typically in fact arise as *Galois groups* or *arithmetic fundamental groups* of schemes] be treated as **abstract topological groups**, rather than as Galois groups or arithmetic fundamental groups [cf. the discussion at the beginning of §3.2; the discussion of Example 3.8.2, (iii), (iv); the discussion of Example 3.8.3, (vi)]. That is to say, to treat these topological groups as Galois groups or arithmetic fundamental groups requires the use of the **ring/scheme structures** involved, i.e., the use of structures which are *not available* since they are **not common/coric** to the rings/schemes that appear on *opposite sides* of the $\mathfrak{log}$-/Θ-link [cf. the discussion of [Alien], §2.10; [IUTchIII], Remarks 1.1.2, 1.2.4, 1.2.5; [IUTchIV], Remarks 3.6.1, 3.6.2, 3.6.3]. In this context, it is also of fundamental importance to observe that it is precisely because these topological groups must be treated as *abstract topological groups* that **anabelian** results play a *central role* in inter-universal Teichmüller theory.

One consequence of the constraint [discussed above] that one must typically work, in inter-universal Teichmüller theory, with structures that are substantially *weaker* than ring structures is the necessity, in inter-universal Teichmüller theory, of allowing for various **indeterminacies**, such as (Ind1), (Ind2), (Ind3), that are somewhat more involved than the relatively simple inner automorphism indeterminacies that occur in [SGA1]. Here, we recall that from the discussion of ($\wedge(\vee)$-Chn) in §3.7 that

> it is precisely the *numerous indeterminacies* that arise in inter-universal Teichmüller theory that give rise to the numerous **logical OR relations "$\vee$"** in the display of ($\wedge(\vee)$-Chn).

On the other hand, once one takes such indeterminacies into account, i.e.,

> once one consents to work with various objects "up to certain suitable indeterminacies" — e.g., by means of **poly-morphisms**, as discussed in §3.7 — it is natural to **identify**, by applying **(ExtInd2)** [as discussed in §3.6], objects that are related to one another by means of collections of isomorphisms [i.e., poly-isomorphisms] that are uniquely determined up to suitable indeterminacies.

Here, we observe that this sort of **(ExtInd2) identification** that occurs repeatedly in inter-universal Teichmüller theory [cf. the discussion of §3.6] may at first glance appear somewhat *novel*. In fact, however, from the point of view of *mathematical foundations* — i.e., just as in the discussion of *inter-universality* given above! — this sort of *(ExtInd2) identification* is *qualitatively* very similar to *numerous classical constructions* such as the following:

(AlgCl) the notion of an **algebraic closure** of a field [cf. the discussion of Example 3.8.1], which is *not constrained* to be a *specific set* constructed from the field;

(DrInv) various categorical constructions such as **direct** and **inverse limits** [i.e., such as *fiber products* of schemes] that are defined by means of some sort of *universal property*, and which are *not constrained* to be *specific sets* even when the given direct or inverse systems are specified set-theoretically;

(HomRs) various constructions of **(co)homology modules** in **homological algebra** that depend on the use of *resolutions* that satisfy certain abstract properties, but which are *not constrained* in a *strict set-theoretic sense* even if the original objects resolved by such resolutions are specified set-theoretically.

That is to say, in each of the *classical constructions*, the *"output object"* is, strictly speaking, from the point of view of mathematical foundations, *not well-defined as a* **particular set**, but rather as a *collection of sets* [where we note that, typically, this "collection" is *not a set*!] that are related to one another — and hence, in common practice, **identified** with one another, in the fashion of (ExtInd2)! — via *unique* [modulo, say, some sort of well-defined indeterminacy] *isomorphisms* by means of some sort of *"universal" property*.

In this context, it is also important to note that, from a foundational point of view, the sort of *"(sub)quotient"* obtained by applying (ExtInd2) [cf. the discussion of *"(sub)quotients"* in (sQLTL) and indeed throughout §3.6] must be regarded, *a priori*, as a **formal (sub)quotient**, i.e., as some sort of *diagram of arrows*. That is to say, at least from an *a priori* point of view,

(NSsQ) any explicit construction of a **"naive set-theoretic (sub)quotient"** necessarily requires the use of some sort of **set-theoretic enumeration** of each of the individual [set-theoretic] objects that are identified, up to isomorphism, via an application of (ExtInd2). On the other hand, as is well-known, typically such set-theoretic enumerations — which often reduce, roughly speaking, to consideration of the *"set of all sets"*! — lead immediately to a *contradiction*.

Indeed, it is precisely this aspect of the constructions of inter-universal Teichmüller theory that motivated the author to include the discussion of **species** in [IUTchIV], §3.

Finally, we recall — cf. also the discussion of §3.10 [especially, (Stp7)] below — that

(LVsQ) it is only in the *final portion* of inter-universal Teichmüller theory, i.e., once one obtains a *formal (sub)quotient* that forms a **"closed loop"**, that one may pass from this *formal (sub)quotient* to a *"coarse/set-theoretic (sub)quotient"* by taking the **log-volume**

[cf. the discussion of [Alien], §3.11, (v); [IUTchIII], Remark 3.9.5, (ix); Steps (x), (xi) of the proof of [IUTchIII], Corollary 3.12].

§3.9. Passage and descent to underlying structures

One fundamental aspect of inter-universal Teichmüller theory lies in the use of numerous **functorial algorithms** that consist of the construction

$$\textit{input data} \quad \rightsquigarrow \quad \textit{output data}$$

of certain *output data* associated to given *input data*. When one applies such functorial algorithms, there are *two ways* in which the output data may be treated [cf. [Alien], §2.7, (iii); the discussion of *"post-anabelian structures"* in [IUTchII], Remark 1.11.3, (iii), (v); [IUTchIII], Remark 1.2.2, (vii)] :

(UdOut) One may consider the *output data* **independently** of the given *input data* and *functorial algorithms* used to construct the output data. In this case, the output data may be regarded as a sort of **"underlying structure"** associated to the input data.

(InOut) One may consider the *output data* as data **equipped** with the additional structure constituted by the *input data*, together with the *functorial algorithm* that gave rise to the output data by applying the algorithm to the input data.

Typical examples of this phenomenon in inter-universal Teichmüller theory are the following [cf. the notational conventions of [IUTchI], Definition 3.1, (e), (f)]:

(sQGOut) *Functorial algorithms that associate to* $\Pi_{\underline{v}}$ *[where* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*] some* **subquotient group** *of* $\Pi_{\underline{v}}$*, such as, for instance, the quotient* $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$: In this sort of situation, treatment of the output data [i.e., subquotient group of $\Pi_{\underline{v}}$] according to (InOut) is indicated by a "$(\Pi_{\underline{v}})$" following the notation for the particular subquotient under consideration; by contrast, treatment of the output data [i.e., subquotient group of $\Pi_{\underline{v}}$] according to (UdOut) is indicated by the *omission* of this "$(\Pi_{\underline{v}})$".

(MnOut) *Functorial algorithms that associate to* $\Pi_{\underline{v}}$ *[where* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*] some sort of [abelian]* **monoid** *equipped with a continuous action by* $\Pi_{\underline{v}}$*, such as, for instance, [data isomorphic to] various subquotient monoids [i.e., "*$\mathcal{O}^{\triangleright}$*", "*$\mathcal{O}^{\times}$*", "*$\mathcal{O}^{\times\boldsymbol{\mu}}$*", etc.] of the multiplicative monoid* $\overline{F}_{\underline{v}}^{\times}$: In this sort of situation, treatment of the output data [i.e., monoid equipped with an action by $\Pi_{\underline{v}}$] according to (InOut) is indicated by a "$(\Pi_{\underline{v}})$" following the notation for the particular monoid equipped with an action by $\Pi_{\underline{v}}$ under consideration; by contrast, treatment of the output data [i.e., monoid equipped with an action by $\Pi_{\underline{v}}$] according to (UdOut) is indicated by the *omission* of this "$(\Pi_{\underline{v}})$".

(PSOut) *Functorial algorithms that associate some sort of* **prime-strip** *to some sort of input data*: In this sort of situation, treatment of the output data [i.e., some sort of prime-strip] according to (InOut) is indicated by a "$(-)$" [where "$-$" is the given *input data*] following the notation for the particular prime-strip under consideration; by contrast, treatment of the output data [i.e., some sort of prime-strip] according to (UdOut) is indicated by the *omission* of this "$(-)$".

Perhaps the most central example of (PSOut) in inter-universal Teichmüller theory is the notion of the **"*q*-/Θ-intertwinings"** on an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* [cf. the discussion of [Alien], §3.11, (v); [IUTchIII], Remark 3.9.5, (viii), (ix); [IUTchIII], Remark 3.12.2, (ii)]:

(ItwOut) This terminology refers to the treatment of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip according to (InOut), relative to the functorial algorithm for constructing the ***q*-pilot** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip [in the case of the "q-intertwining"] or the **Θ-pilot** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip [in the case of the "Θ-intertwining"] from some $\Theta^{\pm\mathrm{ell}}$*NF- or* $\mathcal{D}$*-*$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater*.

In any situation in which one considers a construction from the point of view of (UdOut) — that is to say, as a construction that produces *"underlying data"* [i.e., *"output data"*] from *"original data"* [i.e., *"input data"*]

$$\begin{tikzcd}
\textit{input data} \arrow[r, squiggly] \arrow[d, equal] & \textit{output data} \arrow[d, equal] \\
\textit{original data} & \textit{underlying data}
\end{tikzcd}$$

— it is natural to consider the issue of **descent** to [a functorial algorithm in] the *underlying data* of a **functorial algorithm** in the *original data*. Here, we say that

> a *functorial algorithm* Φ in the *original data* **descends** to a *functorial algorithm* Ψ in the *underlying data* if there exists a functorial isomorphism
>
> $$\Phi \quad \xrightarrow{\sim} \quad \Psi|_{original\ data}$$
>
> between Φ and the *restriction* of Ψ, i.e., relative to the given construction *original data* $\rightsquigarrow$ *underlying data.*

That is to say, roughly speaking, to say that the functorial algorithm Φ in the original data *descends* to the *underlying data* means, in essence, that although the construction constituted by Φ depends, *a priori*, on the **"finer"** *original data*, in fact, up to natural isomorphism, it only depends on the **"coarser"** *underlying data.*

One elementary example of the phenomenon of *descent* may be seen in the situation discussed in (HomRs) in §3.8:

(HmDsc) The various constructions of **(co)homology modules** in **homological algebra** are, strictly speaking, constructions that require as *input data* not just some *given module* [whose (co)homology is computed by the construction], but also some sort of *resolution* of the given module that satisfies certain properties. In fact, however, such constructions of (co)homology modules typically **descend**, up to *unique isomorphism*, to constructions whose *input data* consists solely of the *given module.*

Another illustrative elementary example of the phenomenon of *descent* is the following:

<u>Example 3.9.1</u>: Categories of open subschemes. Let X be a *scheme*, T a *topological space*. Write

- $|X|$ for the *underlying topological space* of X,
- $\text{Open}(X)$ for the category of *open subschemes* of X and *open immersions* over X,
- $\text{Open}(T)$ for the category of *open subsets* of T and *open immersions* over T.

Then the *functorial algorithm*

$$X \quad \mapsto \quad \text{Open}(X)$$

— defined, say, on the category of schemes and morphisms of schemes — is easily verified to *descend*, relative to the construction $X \rightsquigarrow |X|$, to the *functorial algorithm*

$$T \quad \mapsto \quad \text{Open}(T)$$

— defined, say, on the category of topological spaces and continuous maps of topological spaces. That is to say, one verifies immediately that there is a *natural functorial isomorphism*

$$\mathrm{Open}(X) \quad \stackrel{\sim}{\to} \quad \mathrm{Open}(|X|)$$

[i.e., in this case, following the conventions employed in inter-universal Teichmüller theory, a *natural functorial isomorphism class of equivalences of categories* — cf. the discussion of "Monoids and Categories" in [IUTchI], §0].

On the other hand, perhaps the *most fundamental example*, in the context of inter-universal Teichmüller theory, of this phenomenon of *descent* is the following [cf. the notational conventions of [IUTchI], Definition 3.1, (e), (f)]:

(MnDsc) The *topological multiplicative monoid* determined by the *topological ring* given by [the union with $\{0\}$ of] $\mathcal{O}^{\triangleright}(\Pi_X)$ [cf. [Alien], Example 2.12.3, (iii)] — that is to say, a construction that, *a priori*, from the point of view of [AbsTopIII], Theorem 1.9; [AbsTopIII], Corollary 1.10, is a functorial algorithm in the *topological group*

$$\Pi_X$$

[i.e., "$\Pi_{\underline{v}}$", from the point of view (sQGOut)] — in fact **descends** [cf. the discussion at the beginning of [Alien], §2.12; the discussion of [Alien], Example 2.12.3, (i)], relative to passage to the *underlying quotient group* discussed in (SQGOut), to a functorial algorithm in the *topological group*

$$G_k$$

[i.e., "$G_{\underline{v}}$", from the point of view (sQGOut)].

Finally, we remark that often, in inter-universal Teichmüller theory, the output data of the functorial algorithm Φ of the above discussion is regarded *"stack-theoretically"*. That is to say, the output data is *not a single "set-theoretic object"*, but rather a collection [which is not necessarily a set!] of set-theoretic objects linked by uniquely determined poly-isomorphisms of some sort. Typically, this sort of situation arises when one applies **(ExtInd2)** — cf. the discussion of **(NSsQ)** in §3.8. The most *central example* of this phenomenon in inter-universal Teichmüller theory is the **multiradial algorithm** — and, especially, the portion of the multiradial algorithm that involves the $\mathfrak{log}$-*Kummer-correspondence* and closely related operations of *Galois evaluation* — which plays the role of

> *exhibiting the Frobenius-like* Θ*-pilot as* **one possibility** *within a* **collection of possibilities** *constructed via anabelian algorithms from étale-like data*

[cf. the discussion at the end of §3.6, as well as the discussion of §3.10, §3.11, below]. That is to say, the $\mathfrak{log}$-*Kummer-correspondence* and closely related operations of *Galois evaluation* exhibit the Frobenius-like Θ-pilot as *one possibility* within a *collection of possibilities* constructed via anabelian algorithms from étale-like data

not in a **set-theoretic sense** [i.e., *one possibility/element* contained in a *set of possibilities*], but rather in a *"stack-theoretic sense"*, in accordance with various applications of (ExtInd2) [cf. the discussion at the end of §3.6], i.e., as

> *one* **possibility***, up to* **isomorphism***, within some [not necessarily set-theoretic!] collection of* **possibilities**.

As discussed in (LVsQ) in §3.8, one arrives at a *set-theoretic situation* — i.e., *one possibility/element* contained in a *set of possibilities* — only after one obtains a *"closed loop"*, which allows one to pass to a *"coarse/set-theoretic (sub)quotient"* by taking the **log-volume**.

§**3.10. Detailed description of the chain of logical AND relations**

We begin the present §3.10 with the following well-known and, in some sense, *essentially tautological observation*: Just as every form of data — i.e., ranging from *text files* and *webpages* to *audiovisual data* — that can be processed by a computer can, ultimately, be expressed as a [perhaps very long!] *chain* of "0's" and "1's", the well-known *functional completeness*, in the sense of *propositional calculus*, of the collection of *Boolean operators* consisting of *logical AND "$\wedge$"*, *logical OR "$\vee$"*, and *negation "$\neg$"* motivates the point of view that one can, in principle, express

> the **essential logical structure** of any **mathematical argument** or **theory** in terms of **elementary logical relations**, i.e., such as **logical AND "$\wedge$"**, **logical OR "$\vee$"**, and **negation** "$\neg$".

Indeed, it is precisely this point of view that formed the *central motivation* and *conceptual starting point* of the exposition given in the present paper.

From the point of view of the correspondence with the terminology and modes of expression that actually appear in [IUTchI-III] and [Alien], the representation given in the present paper of the *essential logical structure* of inter-universal Teichmüller theory in terms of *elementary logical relations*, i.e., such as *logical AND "$\wedge$"* and *logical OR "$\vee$"*, may be understood as follows:

- **Logical AND "$\wedge$"** corresponds to such terms as
  - *simultaneous execution* and
  - *gluing*

  [cf. [IUTchIII], Remark 3.11.1, (ii); [IUTchIII], Remark 3.12.2, (ii), ($\mathrm{c}^{\mathrm{itw}}$), ($\mathrm{f}^{\mathrm{itw}}$); the final portion of [Alien], §3.7, (i); [Alien], §3.11, (iv)].

- **Logical OR "$\vee$"** corresponds to such terms as
  - *indeterminacies*,
  - *poly-morphisms*, and
  - *projection/(sub)quotient/splitting*

  [cf. §3.7; the title of [IUTchIII]; [IUTchIII], Remark 3.9.5, (xiii), (ix); [Alien], §3.11, (v); [Alien], §4.1, (iv)].

Recall the *essential logical structure* of inter-universal Teichmüller theory summarized in ($\wedge$($\vee$)-Chn)

$$\begin{aligned} A \wedge B \quad &= \quad A \wedge (B_1 \vee B_2 \vee \dots) \\ &\Longrightarrow \quad A \wedge (B_1 \vee B_2 \vee \dots \vee B_1' \vee B_2' \vee \dots) \\ &\Longrightarrow \quad A \wedge (B_1 \vee B_2 \vee \dots \vee B_1' \vee B_2' \vee \dots \vee B_1'' \vee B_2'' \vee \dots) \\ &\qquad\qquad \vdots \end{aligned}$$

[cf. the discussion of §3.6, §3.7]. Observe that if the description of the *various "possibilities"* related via "$\vee$'s" in the above displays is *suitably formulated*, i.e., without superfluous overlaps, then in fact these *logical OR "$\vee$'s"* may be understood as *logical XOR "$\dot{\vee}$'s"*, i.e., we conclude the following:

($\wedge$($\dot{\vee}$)-Chn) The *essential logical structure* of inter-universal Teichmüller theory may be summarized as follows:

$$\begin{aligned} A \wedge B \quad &= \quad A \wedge (B_1 \ \dot{\vee}\ B_2 \ \dot{\vee}\ \dots) \\ &\Longrightarrow \quad A \wedge (B_1 \ \dot{\vee}\ B_2 \ \dot{\vee}\ \dots \ \dot{\vee}\ B_1' \ \dot{\vee}\ B_2' \ \dot{\vee}\ \dots) \\ &\Longrightarrow \quad A \wedge (B_1 \ \dot{\vee}\ B_2 \ \dot{\vee}\ \dots \ \dot{\vee}\ B_1' \ \dot{\vee}\ B_2' \ \dot{\vee}\ \dots \ \dot{\vee}\ B_1'' \ \dot{\vee}\ B_2'' \ \dot{\vee}\ \dots) \\ &\qquad\qquad \vdots \end{aligned}$$

Here, we observe the following:

($\wedge$($\dot{\vee}$)-Chn1) The "$\wedge$'s" in the above display

- arise from the **Θ-link**, which may be thought of as a relationship between certain portions of the **multiplicative** structures of the ring structures arising from the *($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters* in the domain and codomain of the *Θ-link* that are **common** [cf. "$\wedge$"!] to these ring structures.

This situation is reminiscent of

- the fact that from the point of view of **Boolean algebras**, "$\wedge$" corresponds to the **multiplicative** structure of the *field* $\mathbb{F}_2$, which may be regarded, via the splitting determined by Teichmüller representatives, as a multiplicative structure that is **common** [cf. "$\wedge$"!] to $\mathbb{Z}$ and $\mathbb{F}_2$ [cf. Example 2.4.6, (iii)], as well as of

- the discussion of [Alien], §3.11, (iv), (2^{and}), concerning the interpretation of the discussion of **crystals** in [Alien], §3.1, (v), (3^{KS}), in terms of the logical relator "$\wedge$", i.e., as objects that may be **simultaneously** interpreted, up to isomorphism, as pull-backs via one projection morphism **and** [cf. "$\wedge$"!] as pull-backs via the other projection morphism.

($\wedge(\dot{\vee})$-Chn2) The "$\dot{\vee}$'s" in the above display may be understood as corresponding to

- various *indeterminacies* that arise mainly from the **$\mathfrak{log}$-Kummer-correspondence**, i.e., from sequences of iterates of the **$\mathfrak{log}$-link**, which may be thought of as a device for constructing **additive** log-shells. The additive structures of the ring structures arising from the *($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters* in the domain and codomain of the Θ-*link* are structures which, unlike the corresponding multiplicative structures, are **not common** [cf. "$\dot{\vee}$"!] to these ring structures in the domain and codomain of the Θ-*link*.

This situation is reminiscent of

- the fact that from the point of view of **Boolean algebras**, "$\dot{\vee}$" corresponds to the **additive** structure of the *field* $\mathbb{F}_2$, which is an additive structure that is **not shared** [cf. "$\dot{\vee}$"!], relative to the splitting determined by Teichmüller representatives, by $\mathbb{Z}$ and $\mathbb{F}_2$ [cf. Example 2.4.6, (iii)], as well as of
- the discussion of [Alien], §3.11, (iv), (2^{and}), concerning the interpretation of the discussion of *crystals* in [Alien], §3.1, (v), (3^{KS}) in terms of the logical relator "$\wedge$", i.e., where we recall that the two pull-backs of the **rank one Hodge subbundle** [cf. [Alien], §3.1, (v), (5^{KS}); the discussion of Hodge structures in [IUTchI], §I2] do **not**, in general, **coincide** [cf. "$\dot{\vee}$"!], but rather differ by an **additive** *"deformation discrepancy"*, namely, the **Kodaira-Spencer morphism**.

($\wedge(\dot{\vee})$-Chn3) Taken together, ($\wedge(\dot{\vee})$-Chn1) and ($\wedge(\dot{\vee})$-Chn2) may be understood as expressing the fact that the "$\dot{\vee}$'s" and "$\wedge$'s" of the above display correspond, respectively, to the **two underlying combinatorial dimensions** — i.e., **addition** and **multiplication** — of a ring or, alternatively, to the **two-dimensional** nature of the **log-theta-lattice** [cf. the discussion of [IUTchIII], Remark 3.12.2, (i); the latter portion of [Alien], §3.3, (ii)]. Thus, these two dimensions may be understood, alternatively, as corresponding to

- the **arithmetically intertwined Θ-link** and **$\mathfrak{log}$-link** of inter-universal Teichmüller theory, which give rise to the **multiradial representation** up to suitable **indeterminacies** [cf. "$\wedge(\dot{\vee})$"!] of the **Θ-pilot**;
- the description given in Example 2.4.6, (iii), of the **carry-addition** operation on the truncated ring of Witt vectors $\mathbb{Z}/4\mathbb{Z}$ in terms of "$\wedge$" and "$\dot{\vee}$" [cf. "$\wedge(\dot{\vee})$"!];

- the **filtered crystal** discussed in [Alien], §3.1, (v), (5^{KS}), where one may think of the *filtration* [i.e., rank one Hodge subbundle] as "being well-defined up to *indeterminacies*" [cf. "$\wedge(\dot{\vee})$"!], i.e., up to a *"discrepancy"*, which is given by the **Kodaira-Spencer morphism**.

($\wedge(\dot{\vee})$-Chn4) The two dimensions discussed in ($\wedge(\dot{\vee})$-Chn3) may be understood as corresponding to the two dimensions — i.e.,

- the **successive iterates** of the **Frobenius morphism** in **positive characteristic** and
- **successive extensions** to **mixed characteristic**

— of a ring of **Witt vectors** [cf. the discussion of the latter portion of [Alien], §3.3, (ii)]. This relationship to the two dimensions of a ring of **Witt vectors** is *entirely consistent* with the way in which truncated rings of Witt vectors occur in the discussion of Example 2.4.6, (iii), i.e., with the expression

$$\ddot{\vee} \;=\; (\wedge, \dot{\vee})$$

of *mixed characteristic* **"carry-addition"** as a sort of **"intertwining"** between addition and multiplication in the *field* $\mathbb{F}_2$ obtained by *"stacking"* **multiplication** *"$\wedge$" on top of* **addition** *"$\dot{\vee}$"*.

Moreover, in this context, we note that the various correspondences observed in ($\wedge(\dot{\vee})$-Chn3) and ($\wedge(\dot{\vee})$-Chn4) are particularly *fascinating* in that they assert that the **"arithmetic intertwining"** in a ring between **addition** and **multiplication** — i.e., the **mathematical structure** which is in some sense the main object of study in inter-universal Teichmüller theory — may be elucidated by means of a theory [i.e., inter-universal Teichmüller theory!] whose **essential logical structure**, when written symbolically in terms of **Boolean operators** such as "$\wedge$" and "$\dot{\vee}$", amounts precisely to the description [cf. the discussion of ($\ddot{\vee} = \wedge\ \dot{\vee}$) in Example 2.4.6, (iii)] of the **"Boolean intertwining"** that appears in Boolean carry-addition "$\ddot{\vee}$" between Boolean addition "$\dot{\vee}$" and Boolean multiplication "$\wedge$". Put another way, it is as if

(TrHrc) the *arithmetic intertwining* which is the main object of study in inter-universal Teichmüller theory somehow *"induces"/"is reflected in"* a sort of **"structural carry operation"**, or **"trans-hierarchical similitude"**, to the *Boolean intertwining* that constitutes the *essential logical structure* of the theory [i.e., inter-universal Teichmüller theory] that is used to describe it:

$$\textbf{arithmetic intertwining} \quad \rightsquigarrow \quad \textbf{Boolean intertwining}!$$

Finally, we observe that it is also interesting to note that the *essential mechanism* underlying the **Kummer theory** of **theta functions** — which plays a central role in inter-universal Teichmüller theory, i.e., in inducing the *trans-hierarchical similude* discussed in (TrHrc) — namely, the correspondence

$$\begin{array}{rcl} \text{Kummer theory of } \textit{theta functions} & \longleftrightarrow & \textit{one} \text{ valuation/cusp} \\ \text{Kummer theory of } \textit{algebraic rational functions} & \longleftrightarrow & \textit{multiple} \text{ valuations/cusps} \end{array}$$

discussed in [IUTchIII], Remark 2.3.3, (viii), (ix), may itself be thought of as a sort of **trans-hierarchical similitude** between **number fields/local fields** and **function fields** over number fields/local fields.

At a more technical level, the *essential logical structure* of inter-universal Teichmüller theory summarized symbolically in $(\wedge(\dot{\vee})$-Chn) may be understood as consisting of the following steps:

(Stp1) **$\mathfrak{log}$-Kummer-correspondence and Galois evaluation**: This step consists of

> *exhibiting the* **Frobenius-like Θ-pilot** *at the lattice point* $(0, 0)$ *of the log-theta-lattice — i.e., the data that gives rise to the* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip that appears in the* **domain** *of the* **Θ-link** *— as* **one possibility** *within a* **collection of possibilities** *[cf. (ExtInd1)!] constructed via anabelian algorithms from* **holomorphic** *[relative to the* 0*-column]* **étale-like data**.

In this context, it is perhaps worth mentioning that it is a *logical tautology* that the content of the above display may, equivalently, be phrased as follows: this step consists of

> *the* **negation "$\neg$"** *of the assertion of the* **non-existence** *of the* **Frobenius-like Θ-pilot** *at the lattice point* $(0, 0)$ *of the log-theta-lattice — i.e., the data that gives rise to the* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip that appears in the* **domain** *of the* **Θ-link** *— within the* **collection of possibilities** *constructed via certain anabelian algorithms from* **holomorphic** *[relative to the* 0*-column]* **étale-like data**

[cf. also the discussion of (RcnLb) below]. At the level of *labels* of lattice points of the log-theta-lattice, this step corresponds to the *descent operation*

$$(0, 0) \quad \rightsquigarrow \quad (0, \circ)$$

[cf. the discussion at the end of §3.6; the discussion at the end of §3.9; [IUTchIII], Remark 3.9.5, (viii), (sQ1), (sQ2); [IUTchIII], Theorem 3.11, (ii), (iii)]. Finally, we recall that this step already involves the introduction of the **(Ind3) indeterminacy**, which may be understood as a sort of **coarse algorithmic approximation** of the complicated apparatus constituted by the $\mathfrak{log}$-*Kummer-correspondence* and *Galois evaluation* [cf. (ExtInd1), as well as the discussion of ($\mathfrak{log}$ORInd) in §3.11 below; the discussion of the **algorithmic parallel transport (APT)** property in [IUTchIII], Remark 3.11.1, (iv)].

(Stp2) **Introduction of (Ind1)**: This step consists of observing that

> *the anabelian construction algorithms of (Stp1) in fact* **descend** *to — i.e., are* **equivalent** *to algorithms that only require as input data the* **weaker data** *constituted by [cf. the discussion of "descent" in §3.9] — the associated* **mono-analytic étale-like** *data, i.e., in the notation of (sQGOut), the "$G_{\underline{v}}$'s".*

At the level of *labels* of lattice points of the log-theta-lattice, this step corresponds to the *descent operation*

$$(0, \circ) \quad \rightsquigarrow \quad (0, \circ)^{\vdash}$$

[cf. [IUTchIII], Remark 3.9.5, (viii), (sQ1), (sQ2); [IUTchIII], Theorem 3.11, (i), as well as the references to [IUTchIII], Theorem 3.11, (i), in [IUTchIII], Theorem 3.11, (iii)]. Finally, we recall that this step involves the introduction of the **(Ind1) indeterminacy**, which [*very mildly*! — cf. the discussion of (Ind3>1+2) in §3.11 below] increases the *collection of possibilities* under consideration [cf. (ExtInd1)].

(Stp3) **<u>Introduction of (Ind2)</u>**: This step consists of observing that

> *the anabelian construction algorithms of (Stp2) in fact* **descend** *to — i.e., are* **equivalent** *to algorithms that only require as input data the* **weaker data** *constituted by [cf. the discussion of "descent" in §3.9] — the associated* **mono-analytic Frobenius-like** *data, i.e., in the notation of (sQGOut) and (MnOut), the "$G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$'s".*

[That is to say, one constructs log-shells, for instance, as submonoids of "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$", as opposed to subquotients of "$G_{\underline{v}}$".] At the level of *labels* of lattice points of the log-theta-lattice, this step corresponds to the *descent operation*

$$(0, \circ)^{\vdash} \quad \rightsquigarrow \quad (0, 0)^{\vdash}$$

[cf. [IUTchIII], Remark 3.9.5, (viii), (sQ1), (sQ2); [IUTchIII], Theorem 3.11, (i), as well as the references to [IUTchIII], Theorem 3.11, (i), in [IUTchIII], Theorem 3.11, (iii)]. Since the **Θ-link** may be thought of as a sort of *equivalence of labels*

$$(0, 0)^{\vdash} \quad \Longleftrightarrow \quad (1, 0)^{\vdash}$$

— i.e., corresponding to the *full poly-isomorphism* of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips constituted by the Θ-link — this *descent operation* means that the algorithm under consideration may be regarded as an algorithm whose **input data** is the **mono-analytic Frobenius-like** data $(1, 0)^{\vdash}$ arising from the **codomain** of the Θ-link. This step involves the introduction of the **(Ind2) indeterminacy**, which [*very mildly*! — cf. the discussion of (Ind3>1+2) in §3.11 below] increases, at least from an *a priori* point of view, the *collection of possibilities* under consideration [cf. (ExtInd1)]. Finally, we recall that this step plays the important role of

> **isolating** the **$\mathfrak{log}$-link** indeterminacies in the **domain** [i.e., the (Ind3) indeterminacy of (Stp1)] and the **codomain** [i.e., the *log-shift adjustment* discussed in (Stp7) below] of the **Θ-link** from one another

[cf. the discussion of [IUTchIII], Remark 3.9.5, (vii), (Ob7-2); [Alien], §3.6, (iv)]. Here, we recall [cf. the discussion of the final portion of [Alien], §3.3, (ii)] that these $\mathfrak{log}$-*link* indeterminacies on either side of the Θ-link may be understood, in the context of the discussion of (InfH) in §3.3, as corresponding to the copies "$\mathbb{C}^\times$" on either side of the double coset space "$\mathbb{C}^\times \backslash GL_2^+(\mathbb{R})/\mathbb{C}^\times$".

(Stp4) **<u>Passage to the holomorphic hull</u>**: The passage from the collection of *possible regions* that appear in the *output data* of (Stp3) to the collection of regions contained in the **holomorphic hull** — relative to the 1-*column* of the log-theta-lattice — of the union of possible regions of the output data of (Stp3) [cf. [IUTchIII], Remark 3.9.5, (vi); [IUTchIII], Remark 3.9.5, (vii), (Ob5); [IUTchIII], Remark 3.9.5, (viii), (sQ3)] is a simple, straighforward application of (ExtInd1), that is to say, of *increasing the set of possibilities* [i.e., of "$\dot{\vee}$'s"]. The purpose of this step, together with (Stp5) below, is to pass from *arbitrary regions* to regions corresponding to *arithmetic vector bundles* [cf. [IUTchIII], Remark 3.9.5, (vii), (Ob1), (Ob2)].

(Stp5) **<u>Passage to hull-approximants</u>**: This step consists of passing from the collection of arbitrary regions contained in the *holomorphic hull* of (Stp4) to **hull-approximants**, i.e., regions that have the *same global log-volume* as the original "arbitrary regions", but which correspond to *arithmetic vector bundles* [cf. [IUTchIII], Remark 3.9.5, (vii), (Ob6); [IUTchIII], Remark 3.9.5, (viii), (sQ3)]. This operation does *not affect* the logical "$\wedge/\vee$" structure of the algorithm since this operation of passing to hull-approximants does

> **not affect** the collection of **possible value group portions** — i.e., *"$\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips"* — of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips determined by forming the *log-volume* of these regions

[cf. the discussion of [IUTchIII], Remark 2.4.2; the discussion of (IPL) in [IUTchIII], Remark 3.11.1, (iii)].

(Stp6) **<u>Passage to a suitable positive rational tensor power of the determinant</u>**: This step consists of passing from the [regions corresponding to] *arithmetic vector bundles* obtained in (Stp4), (Stp5) to a *suitable tensor power root* of a *tensor power* of the *determinant arithmetic line bundle* of such an arithmetic vector bundle [cf. [IUTchIII], Remark 3.9.5, (vii), (Ob3), (Ob4); [IUTchIII], Remark 3.9.5, (viii), (sQ3)]. Just as in the case of (Stp5), this operation does *not affect* the logical "$\wedge/\vee$" structure of the algorithm since this operation of passing to a suitable positive rational tensor power of the determinant does

> **not affect** the collection of **possible value group portions** —

> i.e., *"$\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips"* — of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips determined by forming the *log-volume* of these regions

[cf. the discussion of [IUTchIII], Remark 2.4.2; the discussion of (IPL) in [IUTchIII], Remark 3.11.1, (iii)].

(Stp7) **<u>Log-shift adjustment</u>**: The *arithmetic line bundles* that appear in (Stp6) occur with respect to the arithmetic holomorphic structure — i.e., in effect, *ring structure* — at the label $(1, 1)$ of the log-theta-lattice, i.e., at a label *vertically shifted* by $+1$ relative to the label $(1, 0)$ that forms the codomain of the Θ-link [cf. the discussion of [IUTchIII], Remark 3.9.5, (vii), (Ob8); [IUTchIII], Remark 3.9.5, (viii), (sQ4)]. That is to say, by applying the algorithm discussed in (Stp1) $\sim$ (Stp6) at *each lattice point* $(1, m)$ [where $m \in \mathbb{Z}$] of the 1-column of the log-theta-lattice, we obtain algorithms with *input data* at $(1, m)$ and *output data* at $(1, m + 1)$ — cf. the *diagonal arrows* of the diagram below. Thus,

> the **totality** of all of these *diagonal arrows* may be thought of as a sort of **endomorphism** of the 1-column of the log-theta-lattice, i.e., an algorithm whose **input data** is the 1-column of the log-theta-lattice, and whose **output data** lies in the **same** 1-column of the log-theta-lattice.

$$\begin{tikzcd}
\text{1-column} & & \text{1-column} \\
\vdots & & \vdots \\
\bullet & & \bullet \\
\bullet \arrow[u, "\mathfrak{log}"'] \arrow[urr] & & \bullet \arrow[u, "\mathfrak{log}"'] \\
\bullet \arrow[u, "\mathfrak{log}"'] \arrow[urr] & & \bullet \arrow[u, "\mathfrak{log}"'] \\
\bullet \arrow[u, "\mathfrak{log}"'] \arrow[urr] & & \bullet \arrow[u, "\mathfrak{log}"'] \\
\vdots & & \vdots
\end{tikzcd}$$

Indeed, one may think of the *input data* of the algorithm discussed in (Stp1) $\sim$ (Stp6) applied at the lattice point $(1, m)$ as being equipped with the *label* $(1, m)^{\Rsh}$, while the *output data* of this algorithm applied at the lattice point $(1, m)$ as being equipped with the *label* ${}^{\Rsh}(1, m + 1)$, i.e., where one regards these labels $(1, m)^{\Rsh}$ and ${}^{\Rsh}(1, m + 1)$ as being *"refinements"* of the respective original labels $(1, m)$ and $(1, m + 1)$ of the 1-column, which may be recovered by *forgetting* the additional data "$\Rsh$" constituted by the input/output of the algorithm under consideration. Moreover, one may consider a *"composite refinement"* ${}^{\Rsh}(1,m)^{\Rsh}$, i.e., obtained by taking the [q-pilot $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip with the *same label* as the] *output data* of ${}^{\Rsh}(1,m)$ as the *input data* of $(1,m)^{\Rsh}$. Here, we note that these *input/output labels* "$\Rsh$" are, in effect, *implicit* in the *species-theoretic* sense [cf. the discussion surrounding (NSsQ) in §3.8; the discussion of [IUTchIV], §3] — where we observe that the *"package of data"* constituted by a *species* may be understood as a sort of *label*! — in which the terms *"input data"/"output data"* are used throughout [IUTchIII] in the discussion of the *multiradial algorithm* of [IUTchIII], Theorem 3.11. Then considering the *totality* of all the *diagonal arrows* corresponds to considering the *"diagonal"* [i.e., in the sense of *"symmetrized"*] collection of data, relative to the **symmetry** given by $\mathbb{Z} \ni m \mapsto m+1 \in \mathbb{Z}$, which induces **compatible symmetries**

$${}^{\Rsh}(1,m)^{\Rsh} \mapsto {}^{\Rsh}(1,m+1)^{\Rsh}; \quad (1,m) \mapsto (1,m+1)$$

[cf. the discussion of (Stp8) below]. Here, it is important to observe that the use of these labels $(1,-)^{\Rsh}$, ${}^{\Rsh}(1,-)$, ${}^{\Rsh}(1,-)^{\Rsh}$ renders explicit the sense in which

> the **gluing**, up to suitable **indeterminacies**, arising from the [algorithm denoted by the] corresponding *diagonal arrow* of the diagram shown above between the *input data* $(1,m)^{\Rsh}$ and the *output data* ${}^{\Rsh}(1,m+1)$ is — **not** (!) a gluing **embedded** in some *familiar ambient space* [cf. the discusssion of (FxEuc), (FxFld) in the final portion of §3.1], but rather — a **formal, diagram-theoretic gluing** between data with **distinct labels** [i.e., induced, in effect, by the *labels* constituted by the various *coordinates* of the *log-theta-lattice*, via the various *descent operations* that appear in the algorithm discussed in (Stp1) $\sim$ (Stp6)],

hence, in particular,

> does **not** give rise to any **nontrivial set-theoretic conclusions** — i.e., such as the **manifest contradiction** (!) that arises, if one *arbitrarily eliminates/forgets* the input/output labels "$\Rsh$", between the *gluing* constituted by the *diagonal arrows* of the diagram shown above [i.e., between the *local value group* portions of the *q-pilot codomain $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips* and *domain Θ-pilot $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips* of the arrows, where the latter is subject to suitable indeterminacies] and the *gluings* of *adjacent local value groups/unit groups* constituted by the $\mathfrak{log}$-*links* — at any of the **intermediate steps** that appear in the course of the execution of (Stp1) $\sim$ (Stp6).

That is to say, it is only by applying the **symmetrization** procedure described above that we obtain a **closed loop** [cf. the discussion of Example 3.1.1, (iii); the discussion below of (DstMp), (FxGl), (NoCmpIss), (Englf); the discussion of [IUTchIII], Remark 3.9.5, (ix); [Alien], §3.11, (v)], i.e., in the language of the discussion surrounding (DltLb) in §3.11 below, a situation that **simulates** — via the introduction of suitable **indeterminacies** [cf. the discussion of (Stp8) below] — a situation in which the

distinct labels on the domain and codomain of the Θ-link have been **eliminated**, hence allows one to draw, in an *essentially formal manner* [cf. the discussion of (Stp8) below], **nontrivial set-theoretic conclusions** [cf. the discussion surrounding (NSsQ), (LVsQ) in the final portion of §3.8]. Here, we note that the *diagonally symmetrized* local value groups that one must consider in order to obtain such a *closed loop* are, in effect, obtained by pulling back these local value groups to the labels "${}^{\Rsh}(1, m)^{\Rsh}$" from the original labels "$(1, m)$" of the 1-column of the log-theta-lattice, i.e., via the *forgetting operation* ${}^{\Rsh}(1, m)^{\Rsh} \rightsquigarrow (1, m)$, hence, in particular, are necessarily **subject** to the condition of **compatibility** with the gluings of *adjacent local value groups/unit groups* constituted by the $\mathfrak{log}$*-links*. This *compatibility* is established by passing to **log-volumes** and applying the **invariance** of the **log-volume** with respect to the **$\mathfrak{log}$-link** [cf. the discussion of [IUTchIII], Remark 3.9.5, (vii), (Ob9)], which may be interpreted as asserting, in effect, that the $\mathfrak{log}$*-link induces*, via passage to the *log-volume*, an *isomorphism* between corresponding adjacent local value groups in the 1-column of the log-theta-lattice [cf. (Stp8) below]. Put another way, this compatibility of the *log-volume* with the $\mathfrak{log}$*-links* in the 1-column of the log-theta-lattice may be regarded as a sort of

> *"version/analogue for iterates of the* $\mathfrak{log}$*-links in the* 1*-column"* of the **saturation** with respect to iterates of the **$\mathfrak{log}$-links** in the 0-column discussed in ($\mathfrak{log}$ORInd) [cf. §3.11] below,

hence, in particular, as a sort of "1-column version/analogue" of the remarkable phenomenon constituted by the "*stark contrast* between the **potency** of [the 0-column] ($\mathfrak{log}$ORInd) and the *utterly meaningless* nature of (ΘORInd)" [cf. the discussion of §3.11 below]. Finally, in this context, it is interesting to observe, from a *historical point of view*, that

> the **set-theoretic confusion** [e.g., in the form the *manifest contradiction* discussed above!] that arises at *intermediate steps* in the course of the execution of (Stp1) $\sim$ (Stp6) if one does *not* take into account the various labels of the *log-theta-lattice*, as well as the additional *input/output labels* discussed above [i.e., which are, in effect, induced by various labels or collections of labels of the log-theta-lattice], is **remarkably reminiscent** of the historical **Weierstrass/Riemann** dispute that arose in complex function theory in the context of the theory of **analytic continuation/Riemann surfaces**, i.e., prior to the development of modern **axiomatic set theory** in the early 20-th century

[cf. the discussion of §1.5] — where we note that, in this analogy, the various labels of the *log-theta-lattice*, as well as the additional *input/output labels* discussed above, correspond to the *set-theoretically distinct labels* of various copies of the *complex open unit disc* that appear in the theory of *analytic continuation/Riemann surfaces.*

(Stp8) <u>**Passage to log-volumes**</u>: The **closed loop** of (Stp7) [cf. also the discussion of Example 3.1.1, (iii); the discussion below of (DstMp), (FxGl), (NoCmpIss), (Englf)] implies that the crucial **logical AND "$\wedge$"** relation *carefully maintained throughout the execution of (Stp1) $\sim$ (Stp7)* yields, upon taking the **log-volume**, a

> **logical AND "$\wedge$"** relationship between the **original *q*-pilot input $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strip** and a certain algorithmically constructed collection of **possible output $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips** within the **same container**, i.e., some copy of the real numbers "$\mathbb{R}$"

[cf. [IUTchIII], Remark 3.9.5, (vii), (Ob9); [IUTchIII], Remark 3.9.5, (viii), (sQ5); [IUTchIII], Remark 3.9.5, (ix); the discussion of Substeps (xi-d), (xi-e) of the proof of [IUTchIII], Corollary 3.12; the discussion of [IUTchIII], Remark 3.12.2, (ii); [Alien], §3.11, (v)]. The **inequality** in the statement of [IUTchIII], Corollary 3.12, then follows as a formal consequence of the **invariance** of the **log-volume** with respect to the **$\mathfrak{log}$-link** [cf. the discussion of the final portion of (Stp7) above; the discussion of Substeps (xi-e), (xi-f), (xi-g) of the proof of [IUTchIII], Corollary 3.12; [IUTchIII], Remark 3.12.2, (iv), (v)]. Here, we observe that the various *indeterminacies* [such as (Ind1), (Ind2), (Ind3)] that arise in the course of applying (Stp1) $\sim$ (Stp7) may be thought of as a sort of *monodromy* associated to the *closed loop* of (Stp7) [cf. also the discussion below of (DstMp), (FxGl), (NoCmpIss), (Englf)]. In this context, we recall from (Stp7) that the *diagram* of arrows "$\nearrow$" from the 1-*column* to the 1-*column* in (Stp7) admits **symmetries**

$$(1, m) \mapsto (1, m + 1)$$

[where $m \in \mathbb{Z}$] that are *compatible* with all of the arrows in the *diagram*, as well as with the various arrows of the $\mathfrak{log}$-*Kummer-correspondence* in the 1-*column*. These *symmetries* allow one to **synchronize** the various **"monodromy indeterminacies"** associated to each "$\nearrow$" [i.e., to each application of (Stp1) $\sim$ (Stp6)], so that one may think in terms of a **single, synchronized** collection of *"monodromy indeterminacies"* associated to the *totality* of "$\nearrow$'s" in (Stp7).

Before proceeding, we pause to examine the meaning of the term **"closed loop"** in (Stp7), (Stp8), which is sometimes a *source of confusion*. The intended meaning of this term is that the sequence of mathematics objects on which the series of operations in (Stp1) $\sim$ (Stp6) [cf. also [IUTchIII], Fig. I.8] are performed forms a *closed loop* in the sense that the ultimate *output data* lies in the same container [i.e., up to a *log-shift* in the 1-column, as discussed in (Stp7)] as the *input data*.

On the other hand, at the level of the actual **mathematical objects** that one is working with, the term *"closed loop"* has the potential to result in certain **fundamental misunderstandings**, since it may be [*mistakenly*!] interpreted as suggesting that

(DstMp) one is considering **two distinct mappings** of *abstract prime-strips* to [Θ, q-]*pilot prime-strips*.

Once one takes this point of view (DstMp), there is inevitably an issue of **diagram commutativity**, i.e., the issue discussed in §3.6, (Syp2), that one must contend with. As discussed in Example 2.4.5, (iv), (v), (vi), (vii), (viii) [cf. also Examples 3.10.1, 3.10.2 below], this point of view (DstMp) corresponds to **EssOR-IUT** [i.e., "essentially OR IUT"], which, as the name suggests, is essentially [thought not precisely!] equivalent to **OR-IUT**, and in particular, constitutes a **fundamental misunderstanding** of the logical structure of inter-universal Teichmüller theory.

Indeed, the **chain of AND relations "$\wedge$"** discussed in §3.6, as well as the present §3.10, which lies at the heart of the *essential logical structure* of inter-universal Teichmüller theory, consists precisely of

(FxGl) **fixing** the Frobenius-like q-pilot at the lattice point $(1, 0)$, as well as the **gluing** [i.e., "$\wedge$"!] via the **Θ-link** of this q-pilot at $(1, 0)$ to the Frobenius-like Θ-pilot at the lattice point $(0, 0)$ [cf. [IUTchIII], Remark 3.12.2, (ii), ($\mathrm{c}^{\mathrm{itw}}$), ($\mathrm{e}^{\mathrm{itw}}$), ($\mathrm{f}^{\mathrm{itw}}$)].

One then proceeds to add to this Θ-pilot at the lattice point $(0, 0)$ *more and more possibilities/indeterminacies* [i.e., "$\vee$", or, alternatively, "$\dot{\vee}$"!] in order to obtain data that **descends** to the same label [i.e., up to a *log-shift* in the 1-column, as discussed in (Stp7)] as the q-pilot at $(1, 0)$. That is to say,

(NoCmpIss) there is **never** any **issue of compatibility** between **two distinct mappings** of abstract prime-strips, as in (DstMp).

From a pictorial point of view, at the level of **mathematical objects**, one is working in (Stp1) $\sim$ (Stp8) — *not with a "closed loop"* (!), but rather — a *single* **fixed** *line segment*

$$\bullet \quad = \quad = \quad \overset{\wedge}{=} \quad = \quad = \quad \bullet$$

corresponding to the **gluing** [i.e., "$\wedge$"!] of (FxGl) [so the "$\bullet$'s" on the *left* and *right* correspond, respectively, to the Θ-pilot at $(0, 0)$ and the q-pilot at $(1, 0)$], which is then subjected to subsequent **"fuzzifications"** [i.e., "$\vee$", or alternatively, "$\dot{\vee}$"!] of the Θ-pilot at $(0, 0)$ [denoted in the following display by the notation "$(\dots)$", which may be thought of as representing a *"fuzzy disc"* that contains the "$\bullet$" on the *left*] that terminate in a situation [cf. the final line of the following display] in which

(Englf) the *fuzzifications* **engulf** the q-pilot at $(1, 0)$, i.e., a situation in which *the distinct labels may be eliminated* and *nontrivial consequences* may be obtained [cf. (Stp7), (Stp8), as well as the discussion surrounding (DltLb) in §3.11 below].

$$\begin{array}{cccccccccccccccc}
 & & & & & & & & \bullet & = & = & \overset{\wedge}{=} & = & = & \bullet & \\
\rightsquigarrow & & & & & & (& \vee & \bullet & =) & = & \overset{\wedge}{=} & = & = & \bullet & \\
\rightsquigarrow & & & & & (& \vee & \vee & \bullet & = & =) & \overset{\wedge}{=} & = & = & \bullet & \\
\rightsquigarrow & & & & (& \vee & \vee & \vee & \bullet & = & = & \overset{\wedge}{=}) & = & = & \bullet & \\
\rightsquigarrow & & & (& \vee & \vee & \vee & \vee & \bullet & = & = & \overset{\wedge}{=} & =) & = & \bullet & \\
\rightsquigarrow & & (& \vee & \vee & \vee & \vee & \vee & \bullet & = & = & \overset{\wedge}{=} & = & =) & \bullet & \\
\rightsquigarrow & (& \vee & \vee & \vee & \vee & \vee & \vee & \bullet & = & = & \overset{\wedge}{=} & = & = & \bullet &)
\end{array}$$

Thus, in summary, throughout the series of *fuzzification* operations constituted by (Stp1) $\sim$ (Stp8), the *line segment* representing the **gluing** [i.e., "$\wedge$"!] of (FxGl) remains **fixed**, so [cf. (NoCmpIss), (Englf)] there is **never** any **issue of compatibility** between **two distinct mappings** of abstract prime-strips, as in (DstMp).

Example 3.10.1: Symmetries as a fundamental non-formal aspect of gluings. One *psychological* aspect, and indeed [in many cases] *possible cause*, of the *fundamental misunderstandings* discussed above [cf. the discussion above surrounding (DstMp), as well as the discussion surrounding (RfsDlg), (DngPrc) in §1.12] concerning the *essential logical structure* of inter-universal Teichmüller theory — i.e., the [*erroneous*!] point of view that this essential logical structure of inter-universal Teichmüller theory should be understood as centering around an issue of **diagram commutativity** — is the following. In any sort of **gluing** situation — i.e., from a category-theoretic point of view, any sort of situation [cf. the numerous examples of gluings discussed throughout the present paper, e.g., in Examples 2.3.2, 2.4.1, 2.4.2, 2.4.3, 2.4.7, 2.4.8, 3.3.1, 3.5.2] in which one considers the [possibly formal] *inductive limit* I of a diagram of the form

$$Y_1 \longleftarrow X \longrightarrow Y_2$$

— there is a certain tendency to fall into the *"mental trap"* of believing that it is a **"tautology"** that

(UnvPrp) **any conceivable nontrivial/interesting property** of I should be understood in terms of the **universal property** of an inductive limit, i.e., the property to the effect that any arrow from I to some object Z should be understood in terms of the issue of **commutativity** of a diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\curvearrowright ?"] & Y_2 \arrow[d] \\ Y_1 \arrow[r] & Z \end{tikzcd}$$

— i.e., where the left-hand vertical and upper horizontal arrows are the arrows in the definition of I.

Here, we observe that

(UnvPrpOR) the *diagram commutativity* issue that arises when one considers the *universal property* discussed in (UnvPrp) is essentially a **logical OR "$\vee$"** situation: that is to say, one can pass from X to Z in the diagram of (UnvPrp) via Y_1 **OR** via Y_2, but there is, *a priori*, *no way of passing* from such an *"OR situation"* to the *desired "AND situation"* that arises when *commutativity holds*, i.e., the situation where one knows that there is a *single arrow* from X to Z that is *simultaneously* equal to the composite of the two arrows that pass through Y_1 *and* equal to the composite of the two arrows that pass through Y_2

— cf. the discussion of Example 2.4.5, (iv), (v), (vi), (vii), (viii). Of course, it *is* indeed a *tautology* that I satisfies a *universal property* as in (UnvPrp), but the *point* is that

(FlsUnv) the fact that I satisfies such a universal property does **not** by any means imply — i.e., as one might *falsely conclude* from the *nuance* carried by the word *"universal"* in ordinary, non-technical contexts! — that **any conceivable nontrivial/interesting property** of I is **best understood** in terms of this **universal property**.

That is to say,

(SymPrp) there are numerous examples in mathematics of objects "I" that are *constructed as gluings*, but that satisfy important nontrivial properties such as **symmetry** properties — e.g., of the sort discussed in §3.2 [cf., especially, Example 3.2.2!] — that do **not** admit *any natural "general nonsense" formulation* in terms of the *universal property* of (UnvPrp).

In the spirit of (UnvPrpOR), it is of interest to note that

(SymPrpAND) **symmetry** properties typically concern some sort of **invariant**, or **"coric"**, structure that is commonly shared throughout various "localizations" of the diagram that gives rise to I, where we recall that it is essentially a *tautology* that this sort of notion of a *commonly shared "coric" property* is a **logical AND "$\wedge$"** situation

— cf. the discussion of Example 2.4.5, (iv), (v), (vi), (vii), (viii), as well as the discussion of (FxGl), (NoCmpIss), (Englf), above. Well-known elementary examples of the phenomenon discussed in (SymPrp) include

(i) the **projective general linear symmetries** [i.e., "PGL_2"] of the **projective line**, which may be constructed as a **gluing** of two copies of the affine line [cf. Example 2.4.7, (iv), (v)];

(ii) the **group structure** of an **elliptic curve**, which, as is well-known, may be constructed as a **gluing** of two cubic planar affine curves;

(iii) the group of **general linear oriented symmetries** $GL_2^+(\mathbb{R})$ of the two-dimensional $\mathbb{R}$-vector space $\mathbb{R}^2$ — cf. the discussion of the *double coset space*

$$\mathbb{C}^\times \backslash GL_2^+(\mathbb{R}) / \mathbb{C}^\times$$

in §3.3, (InfH) — in the situation of the **complex Teichmüller deformations** discussed in Example 3.3.1, where we recall that this situation may be regarded [cf. the discussion at the beginning of Example 3.5.2] as a **"gluing"** of *two distinct copies* of the *complex plane* $\mathbb{C}$ along a *common underlying two-dimensional* $\mathbb{R}$*-vector space* $\mathbb{R}^2$ [cf. also Example 3.5.2, (iii)], and we observe that these *symmetries* $GL_2^+(\mathbb{R})$ allow one to regard each of the two *holomorphic structures* [i.e., copies of $\mathbb{C}$] that appear in this gluing as **indeterminate $GL_2^+(\mathbb{R})$-conjugates** of the subgroup $\mathbb{C}^\times \subseteq GL_2^+(\mathbb{R})$ inside the **common container** $GL_2^+(\mathbb{R})$;

(iv) the **common holomorphic structure** that appears in the classical theory of **analytic continuation** of one-variable complex holomorphic functions — cf., e.g., the discussion of the theory of *analytic continuation* of the **complex logarithm** in the discussion surrounding (FxEuc) in §3.1, as well as in the *historical discussion* of §1.5 and the discussion of the *toral rotations* "$\mathbb{C}^\times$" in §3.3, (InfH) — where we observe that such *analytic* *continuations* may be regarded as **gluings** of copies of the *complex open unit disc*, and that the *common holomorphic structure* may be regarded as a sort of **common symmetry** of such gluings, i.e., if one thinks in terms of *"almost complex structures"*, that is to say, in terms of the *symmetry* of the tangent bundle given by *multiplication by* $i = \sqrt{-1}$.

Here, we observe in passing that it is of interest to note, in the context of (ii), that the *group structure* of an *elliptic curve*, as well as the existence of *invariant* [i.e., with respect to this group structure] *differentials* on the elliptic curve, play a *central role* in the argument discussed in Example 3.2.1 — an argument that itself played a *fundamental role* in motivating the *essential logical structure* of inter-universal Teichmüller theory [cf. Example 3.2.1, (viii)]. Of course, as discussed extensively in §3.3 [cf. (InfH); Examples 3.3.1, 3.3.2], the situations discussed in (iii), (iv) also exhibit *numerous important structural similarities* to various important aspects of inter-universal Teichmüller theory.

__Example 3.10.2__**: Chains of logical AND relations via commutative diagrams.** We maintain the notation of Example 3.10.1.

(i) As discussed at the beginning of Example 3.10.1,

(ORAch) the **"diagram-commutativity"**, or **"OR approach"** [cf. (UnvPrp), (UnvPrpOR)],

$$\begin{tikzcd}
Y_1 \arrow[dr] & X \arrow[l] \arrow[r] \arrow[d, phantom, "\curvearrowright ?"] & Y_2 \arrow[dl] \\
 & Z &
\end{tikzcd}$$

to analyzing the structure of the [possibly formal] *inductive limit* I of the upper line of the above diagram constitutes a sort of **"mental trap"** that appears to be, in many cases, one of the **main causes** of the **fundamental misunderstandings** that exist in certain sectors of the mathematical community concerning inter-universal Teichmüller theory.

[Here, we recall that, in the situation considered in inter-universal Teichmüller theory, the *initial gluing in the first horizontal line* of the diagram of (ORAch) corresponds to the gluing constituted by the **Θ-link** between the *"Θ-pilot object in the Θ-($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater"* Y_1 and the *"q-pilot object in the q-($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater"* Y_2 along the prime-strip data X, while "Z" is to be understood as some sort of *container* that contains both Y_1 and Y_2.] The state of affairs summarized in (ORAch) thus prompts the following question:

(Q∧(∨)CCD) Since the tendency of many mathematicians — especially those who work in abstract areas of *arithmetic geometry*! — to try to interpret the essential logical structure of inter-universal Teichmüller theory in terms of the *[incorrect!]* *"OR approach"* of (ORAch) appears to stem, to a substantial extent, from the fact that such mathematicians often have a strong desire to formulate structural properties of mathematical objects in terms of **commutative diagrams**, is it possible to somehow formulate the **essential logical structure** of inter-universal Teichmüller theory summarized in ($\wedge$($\vee$)-Chn) — or, alternatively, ($\wedge$($\dot{\vee}$)-Chn) — in terms of some sort of *collection of commutative diagrams*?

Here, we recall from the discussion at the beginning of the present §3.10 [cf. also the discussion surrounding (SymIUT) in §1.12] that the *main thrust* of the present paper lies in formulating the *essential logical structure* of inter-universal Teichmüller theory [modulo certain *"blackboxes"* in *anabelian geometry* and the theory of *étale theta functions*] in terms of **logical AND "$\wedge$"/logical OR "$\vee$" relations**. The fundamental motivation for this approach taken in the present paper lies in the point of view that such *logical AND "$\wedge$"/logical OR "$\vee$" relations* constitute the **"most primitive/fundamental/universal"** means available for documenting the essential logical structure of a mathematical argument. This point of view is also closely related to the point of view of *computer verification* of mathematical arguments discussed at the beginning of §1.12 [cf. (CmbVer), (Algor)]. On the other hand, even if this point of view is in some sense *"correct"* from an *abstract, theoretical standpoint*, it is *not necessarily* the case that this point of view is also correct from the somewhat more *practical standpoint* of developing an optimally efficient means of *communicating* the essential logical structure of inter-universal Teichmüller theory to other mathematicians. It is precisely this practical standpoint that motivates the question posed in (Q$\wedge$($\vee$)CCD).

(ii) In a word, it is not very difficult to give an *affirmative* answer to the question posed in (Q$\wedge$($\vee$)CCD). Indeed,

($\wedge$($\vee$)CCD) one may formulate the **essential logical structure** of inter-universal Teichmüller theory summarized in ($\wedge$($\vee$)-Chn) — or, alternatively, ($\wedge$($\dot{\vee}$)-Chn) — in terms of a **chain of [tautologically!] commutative diagrams** as follows:

$$\begin{tikzcd}
Y_1 \arrow[d] \arrow[dr, phantom, "\curvearrowright !"] & X \arrow[l] \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright !"] & Y_2 \arrow[d, equal] \\
Y_1' \arrow[d] \arrow[dr, phantom, "\curvearrowright !"] & X \arrow[l] \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright !"] & Y_2 \arrow[d, equal] \\
Y_1'' & X \arrow[l] \arrow[r] & Y_2 \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

— where the left-hand vertical arrows are natural inclusion morphisms into **larger and larger containers**, and "$\curvearrowright$!" denotes the *tautological commutativity* of the square in question.

That is to say, this *chain of [tautologically!] commutative diagrams* exhibits the *initial gluing in the first horizontal line* — i.e., the gluing constituted by the **Θ-link** between the *"Θ-pilot object in the Θ-($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater"* Y_1 and the *"q-pilot object in the q-($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater"* Y_2 along the prime-strip data X — as an object that maps *tautologically* to the *gluings in the subsequent horizontal lines*, which consist of *copies* of the *initial gluing in the first horizontal line*, but with the "Θ-*pilot object* in the Θ-($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater" Y_1 regarded up to *suitable indeterminacies*, or *"alternative possibilities"* [i.e., as discussed in (Stp1) $\sim$ (Stp8)], exhibited in *larger and larger containers* "Y_1'", "Y_1''", ... Put another way, this approach consists of *embedding* Y_1 *into* **larger and larger containers** [i.e., the containers obtaining by adding the various *indeterminacies*] until the container becomes sufficiently large as to **contain/engulf** *[not only* Y_1*, but also (!)]* Y_2 [cf. the discussion surrounding (Englf)].

(iii) At first glance, the approach discussed in ($\wedge(\vee)$CCD) may appear to some arithmetic geometers to be *entirely unfamiliar* and *fundamentally different* from the approach taken in the numerous theories in arithmetic geometry that existed prior to the appearance of inter-universal Teichmüller theory. In fact, however, the approach discussed in ($\wedge(\vee)$CCD) is entirely analogous to the approach taken in the classical theory of **crystals**. Moreover, this analogy with the classical theory of crystals is *completely compatible* with the discussion of the *strong structural resemblances* between inter-universal Teichmüller theory and the theory of crystals given in [Alien], §3.1, (v) [cf. also §3.5 of the present paper, as well as the discussion of ($\wedge(\dot{\vee})$-Chn) in the present §3.10]. Indeed, the **"container of indeterminacies"** discussed in (ii) may be understood as corresponding to the **PD-envelopes/thickenings** that apppear in the theory of crystals. That is to say, the approach typically taken in the theory of crystals to constructing [nontrivial!] crystals — i.e., to constructing isomorphisms

$$p_1^*(-) \overset{\sim}{\to} p_2^*(-)$$

between the pull-backs via the two natural projections

$$Y \xleftarrow{p_1} Y \times Y \xrightarrow{p_2} Y$$

of some object "$(-)$" on some scheme Y — proceeds **not** by securing some sort of commutative diagram

$$\begin{tikzcd}
Y \arrow[dr] & Y \times Y \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, phantom, "\curvearrowright ?"] & Y \arrow[dl] \\
& Z &
\end{tikzcd}$$

— i.e., corresponding to the *[incorrect!] "OR approach"* of (ORAch)! — such that the object "$(-)$" on Y *descends* to Z, but rather by restricting to the **"sufficiently large container"** constituted by a suitable **PD-envelope/thickening** of the diagonal in $Y \times Y$ and verifying that this container is indeed *sufficiently large* that the restriction to this PD-envelope/thickening of $p_1^*(-)$ already contains, up to isomorphism, the inverse image $p_2^{-1}(-)$.

§3.11. The central importance of the $\mathfrak{log}$-Kummer-correspondence

In the context of the discussion of §3.10, it is important to recall that, whereas (Stp2) $\sim$ (Stp8) are *technically trivial* in the sense that they concern operations that are very elementary and only require a few lines to describe, the **$\mathfrak{log}$-Kummer-correspondence** and **Galois evaluation** operations that comprise (Stp1) depend on the *highly nontrivial* theory of [EtTh] and [AbsTopIII]. Moreover, the technical description of these operations that comprise (Stp1) occupies the bulk of [IUTchI-III]. The central importance of (Stp1) may also be seen in the **subordinate nature** of (Ind1), (Ind2) [which occur in (Stp2), (Stp3)] relative to (Ind3) [which occurs in (Stp1)], i.e., in the sense that

(Ind3>1+2) once one constructs the *output* of the **multiradial representation** of the Θ-pilot [cf. [IUTchIII], Theorem 3.11, (ii)] via **tensor-packets of log-shells** in such a way that each local portion of this output is *stable* with respect to the indeterminacy (Ind3), these local portions of the output are *automatically "essentially stable"* [i.e., stable up to discrepancies at the valuations $\in \underline{\mathbb{V}}^{\mathrm{bad}}$ that affect the resulting log-volumes only up to *very small/essentially negligible order*] with respect to the indeterminacies (Ind1), (Ind2) [cf. [IUTchIII], Theorem 3.11, (i)].

Finally, we observe that this property (Ind3>1+2) is strongly reminiscent of the discussion of (CnfInd1+2) and (CnfInd3) in §3.5.

One way to understand the content of the operations of (Stp1) is as follows. These operations may be regarded as a sort of

($\mathfrak{log}$ORInd) **saturation** of the **Frobenius-like Θ-pilot** at the lattice point $(0,0)$ of the log-theta-lattice — i.e., which is *linked*, via the Θ-link, to the **Frobenius-like $\boldsymbol{q}$-pilot** at $(0,1)$ — with respect to *all of the* **possibilities** that occur in the 0-column of the log-theta-lattice, i.e., all of the possibilities that arise from a *possible confusion* between the *domain* and *codomain* of the **$\mathfrak{log}$-links** in the 0-column [cf. the description of (Stp1)].

In this sense, the content of (Stp1) is *formally reminiscent* of the "(NeuORInd)" that appeared in the discussion of §3.4, i.e., which may be understood as a sort of

(ΘORInd) **saturation** of the **Frobenius-like Θ-pilot** at the lattice point $(0,0)$ of the log-theta-lattice with respect to *all of the* **possibilities** — i.e., "Θ-$\mathfrak{plt}$", "$\mathfrak{q}$-$\mathfrak{plt}$" [cf. (NeuORInd2)] — that arise from a *possible confusion* between the *domain* and *codomain* of the **Θ-link** joining the lattice points $(0,0)$ and $(0,1)$.

[In this context, we note that the *logical OR "$\vee$'s"* that appear in ($\mathfrak{log}$ORInd), (ΘORInd) may in fact be understood as *logical XOR "$\dot{\vee}$'s"* — cf. the discussion surrounding ($\wedge(\dot{\vee})$-Chn) in §3.10.] On the other hand, whereas, as observed in the discussion at the end of §3.4, (ΘORInd) yields a *meaningless/useless* situation that does not give rise to any interesting mathematical consequences, ($\mathfrak{log}$ORInd), by contrast, is a **highly potent technical device** that forms the **technical core** of inter-universal Teichmüller theory.

Before preceding, we observe that, in this context, it is interesting to note that

> *both* of these **"saturation operations"** ($\mathfrak{log}$ORInd) and (ΘORInd) are in some sense **qualitatively similar** to the **label crushing** operation **(ExtInd2)**.

Indeed, (ExtInd2) consists, roughly speaking, of regarding mathematical objects of a certain type *up to isomorphism*, i.e., of **saturating** within an isomorphism class of mathematical objects of a certain type [cf. the discussion of (ExtInd2) in §3.8, as well as the discussion of (DltLb) below].

The *stark contrast* between the **potency** of ($\mathfrak{log}$ORInd) and the *utterly meaningless* nature of (ΘORInd) is *highly reminiscent* of the **central role** played, in

Example 3.3.2, (iv), by *invariance* with respect to

$$\iota \ = \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{C}^{\times} \subseteq GL_2^{+}(\mathbb{R})$$

[where we recall from (InfH) that $\mathbb{C}^{\times}$ corresponds to the **$\mathfrak{log}$-link**!], which lies in stark contrast to the *utterly meaningless* nature of considering invariance with respect to *dilations* $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) \in GL_2^{+}(\mathbb{R})$ [where we recall from (InfH) that such dilations correspond to the Θ-*link*].

One way to witness the potency of ($\mathfrak{log}$ORInd) is as follows. Recall that the **Θ-link**, by definition [cf. [IUTchIII], Definition 3.8, (ii)], consists of

· a **dilation** applied to the *local value group* portions of the ring structures in its domain and codomain, coupled with

· a **full poly-isomorphism** — which *preserves log-volumes*, hence is **non-dilating**! — between the local "$\mathcal{O}^{\times \boldsymbol{\mu}}$'s", i.e., the *local unit group* portions, of these ring structures.

By contrast, the **$\mathfrak{log}$-links** in the 0-column of the log-theta-lattice have the effect of *"juggling/rotating/permuting"* the *local value group* portions and *local unit group* portions of the ring structures that appear in this 0-column [cf., e.g., the discussion of [Alien], Example 2.12.3, (v)]. From this point of view, the *tautologically vertically coric* — i.e., **invariant** with respect to the application of the **$\mathfrak{log}$-link**! — nature of the *output* data of ($\mathfrak{log}$ORInd) is already somewhat **"shocking"** in nature. That is to say, the tautologically vertically coric nature of this output data of ($\mathfrak{log}$ORInd) suggests that

(Di/NDi) this output data already exhibits some sort of **equivalence**, up to perhaps some sort of mild discrepancy, between the **dilated** and **non-dilated** portions of the Θ-link.

Such an *equivalence* already strongly suggests that some sort of **bound** on **heights** should follow as a *formal consequence*, i.e., in the style of the classical argument that implies the *isogeny invariance of heights of elliptic curves* [cf. the discussion of [Alien], §2.3, §2.4, as well as the discussion of Example 3.2.1 in the present paper; the discussion of §3.5 in the present paper].

Finally, we conclude by emphasizing that, in inter-universal Teichmüller theory,

(DltLb) ultimately one *does* want to find some way in which to **delete/eliminate** the **distinct labels** on the Θ- and q-pilot objects [i.e., "Θ-$\mathfrak{plt}$" and "$\mathfrak{q}$-$\mathfrak{plt}$"] in the domain and codomain of the Θ-link

[cf. the discussion of Example 3.1.1, (iii); the discussion of (AOL4), (AOΘ4) in §3.4; (Stp7), (Stp8) in §3.10], that is to say, **not** via the *naive, simple-minded* approach of (ΘORInd) [i.e., (NeuORInd2) in the discussion of §3.4], but rather via the *indirect approach* of applying **descent operations**

$$(0,0) \quad \overset{\text{(Stp1)}}{\rightsquigarrow} \quad (0,\circ) \quad \overset{\text{(Stp2)}}{\rightsquigarrow} \quad (0,\circ)^{\vdash} \quad \overset{\text{(Stp3)}}{\rightsquigarrow} \quad (0,0)^{\vdash} \quad \overset{\text{(Stp3)}}{\Longleftrightarrow} \quad (1,0)^{\vdash}$$

as discussed in (Stp1) $\sim$ (Stp8) [cf., especially, (Stp7), (Stp8)] of §3.10, i.e., an approach that centers around ($\mathfrak{log}$ORInd). This approach is based on the various **anabelian reconstruction algorithms** discussed in (Stp1) $\sim$ (Stp3), which allow one to exhibit the **Frobenius-like Θ-pilot object** at $(0,0)$ as *one possibility* among *some broader collection of possibilities* that arise from the introduction of various types of **indeterminacy**. In this context, we observe [cf. the discussion of (ExtInd2), (NSsQ) at the end of §3.9] that since such anabelian reconstruction algorithms only reconstruct various types of mathematical objects [i.e., monoids/pseudo-monoids/mono-theta environments, etc.] *not "set-theoretically on the nose"* [i.e., *not* in the sense of *strict set-theoretic equality*], but rather *up to [a typically essentially unique, if one allows for suitable indeterminacies] isomorphism*, it is not immediately clear

(RcnLb) in what sense such anabelian reconstruction algorithms yield a **reconstruction** of the **crucial labels** — i.e., such as "$(0,0)$" — that underlie the **crucial logical AND "$\wedge$"** structure discussed in §3.4 [cf., especially, (AOL1), (AOΘ1)].

The point here is that indeed such anabelian reconstruction algorithms are *not capable* of reconstructing such labels *"set-theoretically on the nose"*.

On the other hand, in this context, it is important to recall the *essential substantive content* of the various **labels** involved:

(HolFrLb) $\underline{(0,0)}$: The **holomorphic Frobenius-like** data labeled by $(0,0)$ consists of various monoids/pseudo-monoids/mono-theta environments, etc., regarded as *abstract monoids/pseudo-monoids/mono-theta environments, etc.*, i.e., as objects that are *not* equipped with the auxiliary data of how they might have been reconstructed via anabelian algorithms from *holomorphic étale-like* data labeled $(0,\circ)$ [cf. the discussion of (UdOut), (InOut), (PSOut), (ItwOut) in §3.9]. In particular, such monoids/pseudo-monoids/mono-theta environments, etc., are **not invariant** with respect to the *"juggling/rotating/permuting"* of *local value group* portions and *local unit group* portions effected by the **$\mathfrak{log}$-links** in the 0-column of the log-theta-lattice, but rather correspond to a **temporary cessation** [cf. the label $(0,0)$ as opposed to the label $(0,\circ)$!] of this operation of *juggling/rotation/permutation.*

(MnAlyLb) $\underline{(0,0)^{\vdash}}$: The **mono-analytic Frobenius-like** data labeled by $(0,0)^{\vdash}$ consists of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* determined by the *Frobenius-like* Θ*-pilot* at $(0,0)$, regarded as an *abstract* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* [cf. the discussion of (UdOut), (InOut), (PSOut), (ItwOut) in §3.9]. Thus, the transition of labels

$$(0,0) \quad \rightsquigarrow \quad (0,0)^{\vdash}$$

consists of an operation of **forgetting** some sort of auxiliary structure [cf. the discussion of (UdOut) in §3.9]. Here, we recall that this construction of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip determined by the Frobenius-like Θ-pilot at $(0,0)$ is *technically possible* precisely because of the *"temporary cessation"* discussed above [cf. the discussion of the *definition of the* Θ*-link* in [Alien], §3.3, (ii), as well as in §3.3 of the present paper].

Thus, the *nontrivial substantive content* of the anabelian reconstruction algorithms of (Stp1) $\sim$ (Stp3) — and hence of the **descent operations**

$$(0,0) \quad \overset{(\text{Stp1}) \sim (\text{Stp3})}{\rightsquigarrow} \quad (0,0)^{\vdash}$$

that result from these anabelian reconstruction algorithms — consists of statements to the effect that

(FrgInv) the operation of **forgetting** discussed in (MnAlyLb) can in fact, if one allows for suitable **indeterminacies**, be **inverted**.

It is precisely this **invertibility** (FrgInv), up to suitable *indeterminacies*, of the operation of **forgetting** discussed in (MnAlyLb), together with the fact that

(GluDt) the **only data** appearing in the reconstruction algorithms [i.e., in the 0-*column*] that is **glued** [cf. the discussion of [IUTchIII], Remark 3.11.1, (ii); the final portion of [Alien], §3.7, (i), as well as the discussion of the *(Ind2) indeterminacy* in the final portion of §3.11, (Stp3), in the present paper] to data in the 1-*column* is the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* labeled $(0,0)^{\vdash}$,

that ensures that the *descent operations* discussed above do indeed **preserve** the crucial **logical AND "∧"** relations discussed in §3.4, §3.6, §3.7, §3.10, i.e., even though the reconstruction algorithms underlying these descent operations do *not* yield reconstructions of the various labels "$(0,0)$", etc., "set-theoretically on the nose".

Bibliography

[AnCnCv] Illustration of analytic continuation along a curve, available at the following URL [where we note that when "copy-and-pasting" this URL, typically it will be necessary to *re-enter* the *underscore* character "_" *manually*, as well as to *delete* stray characters that appear as a result of the *carriage return* between the first and second lines of the URL]:
`https://commons.m.wikimedia.org/wiki/File:Analytic_continuation_`
`along_a_curve.png`

[AnCnLg] Illustration of analytic continuation of the complex logarithm, available at the following URL [where we note that when "copy-and-pasting" this URL, typically it will be necessary to *re-enter* the *underscore* character "_" *manually*, as well as to *delete* stray characters that appear as a result of the *carriage return* between the first and second lines of the URL]:
`https://ja.m.wikipedia.org/wiki/%E3%83%95%E3%82%A1%E3%82%A4%E3%`
`83%AB:Imaginary_log_analytic_continuation.png`

[Bns] D. Benois, An introduction to p-adic Hodge theory, *Perfectoid spaces*, *Infosys Sci. Found. Ser.*, Springer-Verlag (2022), pp. 69-219.

[Btt] U. Bottazzini, "Algebraic truths" vs. "geometric fantasies": Weierstrass' response to Riemann, *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, Higher Ed. Press (2002), pp. 923-934.

[EMSCOP] European Mathematical Society, *Code of Practice*, approved by the EMS Executive Committee (October 2012), available at the following URL:
`https://euromathsoc.org/code-of-practice`

[FC] G. Faltings and C.-L. Chai, *Degenerations of Abelian Varieties*, Springer-Verlag (1990).

[FsADT] I. Fesenko, Arithmetic deformation theory via arithmetic fundamental groups and nonarchimedean theta-functions, notes on the work of Shinichi Mochizuki, *Eur. J. Math.* **1** (2015), pp. 405-440.

[GeoSph] Illustration of the metric geometry of a sphere, available at the following URL [where we note that when "copy-and-pasting" this URL, typically it will be necessary to *re-enter* the *underscore* character "_" *manually*, as well as to *delete* stray characters that appear as a result of the *carriage return* between the first and second lines of the URL]:
`https://ja.m.wikipedia.org/wiki/%E3%83%95%E3%82%A1%E3%82%A4%E3`
`%83%AB:Sphere_wireframe_10deg_6r.svg`

[FsDss] I. Fesenko, *About certain aspects of the study and dissemination of Shinichi Mochizuki's IUT theory*, available at the following URL:
`https://ivanfesenko.org/wp-content/uploads/2021/10/rapg.pdf`

[FsPio] I. Fesenko, *On pioneering mathematical research, on the occasion of the announced publication of the IUT papers by Shinichi Mochizuki*, available at the following URL:
`https://ivanfesenko.org/wp-content/uploads/2021/10/rpp.pdf`

[AnHst] E. Hairer and G. Wanner, *Analysis by its history*, Springer-Verlag (2008).

[FKvid] F. Kato, *abc Conjecture and New Mathematics - Prof. Fumiharu Kato, Oct 7, 2017 (with English subtitles)*, video lecture in Japanese (with English subtitles) available at the following URL:
`https://www.youtube.com/watch?v=fNS7N04DLAQ&feature=youtu.be`

[pTch] S. Mochizuki, *Foundations of p-adic Teichmüller Theory*, *AMS/IP Studies in Advanced Mathematics* **11**, American Mathematical Society/International Press (1999).

[HASurI] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves I, *Arithmetic Fundamental Groups and Noncommutative Algebra*, *Proceedings of Symposia in Pure Mathematics* **70**, American Mathematical Society (2002), pp. 533-569.

[AbsAnab] S. Mochizuki, The Absolute Anabelian Geometry of Hyperbolic Curves, *Galois Theory and Modular Forms*, Kluwer Academic Publishers (2004), pp. 77-122.

[EtTh] S. Mochizuki, The Étale Theta Function and its Frobenioid-theoretic Manifestations, *Publ. Res. Inst. Math. Sci.* **45** (2009), pp. 227-349.

[GenEll] S. Mochizuki, Arithmetic Elliptic Curves in General Position, *Math. J. Okayama Univ.* **52** (2010), pp. 1-28.

[AbsTopIII] S. Mochizuki, Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms, *J. Math. Sci. Univ. Tokyo* **22** (2015), pp. 939-1156.

[IUTchI] S. Mochizuki, Inter-universal Teichmüller Theory I: Construction of Hodge Theaters, *Publ. Res. Inst. Math. Sci.* **57** (2021), pp. 3-207.

[IUTchII] S. Mochizuki, Inter-universal Teichmüller Theory II: Hodge-Arakelov-theoretic Evaluation, *Publ. Res. Inst. Math. Sci.* **57** (2021), pp. 209-401.

[IUTchIII] S. Mochizuki, Inter-universal Teichmüller Theory III: Canonical Splittings of the Log-theta-lattice, *Publ. Res. Inst. Math. Sci.* **57** (2021), pp. 403-626.

[IUTchIV] S. Mochizuki, Inter-universal Teichmüller Theory IV: Log-volume Computations and Set-theoretic Foundations, *Publ. Res. Inst. Math. Sci.* **57** (2021), pp. 627-723.

[ExpEst] S. Mochizuki, I. Fesenko, Y. Hoshi, A. Minamide, W. Porowski, Explicit Estimates in Inter-universal Teichmüller Theory, *Kodai Math. J.* **45** (2022), pp. 175-236.

[Rpt2013] S. Mochizuki, *On the verification of inter-universal Teichmüller theory: a progress report (as of December 2013)*, available at the following URL:
`http://www.kurims.kyoto-u.ac.jp/~motizuki/IUTeich%20Verification%20Report%202013-12.pdf`

[Rpt2014] S. Mochizuki, *On the verification of inter-universal Teichmüller theory: a progress report (as of December 2014)*, available at the following URL:
`http://www.kurims.kyoto-u.ac.jp/~motizuki/IUTeich%20Verification%20Report%202014-12.pdf`

[Pano] S. Mochizuki, A Panoramic Overview of Inter-universal Teichmüller Theory, *Algebraic number theory and related topics 2012, RIMS Kōkyūroku Bessatsu* **B51**, Res. Inst. Math. Sci. (RIMS), Kyoto (2014), pp. 301-345.

[Alien] S. Mochizuki, The Mathematics of Mutually Alien Copies: from Gaussian Integrals to Inter-universal Teichmüller Theory, *Inter-universal Teichmuller Theory Summit 2016, RIMS Kōkyūroku Bessatsu* **B84**, Res. Inst. Math. Sci. (RIMS), Kyoto (2021), pp. 23-192.

[Rpt2018] S. Mochizuki, *Report on discussions, held during the period March 15 - 20, 2018, concerning inter-universal Teichmüller theory (IUTch)*, available at the following URL:
`http://www.kurims.kyoto-u.ac.jp/~motizuki/Rpt2018.pdf`

[Dsc2018] Webpage "March 2018 Discussions on IUTeich", available at the following URL:
`http://www.kurims.kyoto-u.ac.jp/~motizuki/IUTch-discussions-2018-03.html`

[Mumf1] D. Mumford, *Abelian Varieties*, Oxford Univ. Press (1974).

[Mumf2] D. Mumford, An Analytic Construction of Degenerating Abelian Varieties over Complete Rings, Appendix to [FC].

[NSW] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, *Grundlehren der Mathematischen Wissenschaften* **323**, Springer-Verlag (2000).

[Pnc] H. Poincaré, translated by F. Maitland, *Science and Method*, Cosimo Classics (2010).

[SGA1] A. Grothendieck et al., *Revêtement étales et groupe fondamental, Séminaire de Géometrie Algébrique du Bois Marie 1960-1961 (SGA1), dirigé par A. Grothendieck, augmenté de deux exposés de M. Raynaud*, *Lecture Notes in Mathematics* **224**, Springer-Verlag (1971).

[Tsjm] S. Tsujimura, Combinatorial Belyi cuspidalization and arithmetic subquotients of the Grothendieck-Teichmüller group, *Publ. Res. Inst. Math. Sci.* **56** (2020), pp. 779-829.

[Ymgt] S. Yamagata, A counterexample for the local analogy of a theorem by Iwasawa and Uchida, *Proc. Japan Acad.* **52** (1976), pp. 276-278.